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ON VARIATIONAL TREATMENT OF NONLINEAR HAMMERSTEIN EQUATIONS

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The theory of integral equations has been developed within our century. A lot of problems in differential equations can be written as integral equations. The last ones have turned out to fit to functional analysis methods and approximation procedures. Thus, since their inception by A. Hammerstein in 1930, the integral equations called after his name, emphasized new classes of operators with peculiar properties. Among them there are angle-bounded mappings, in particular, the symmetric linear operators considered further.

It is our purpose to present, in a simple manner, an operator approach of classical problems for semilinear elliptic equations.

1. NONLINEAR HAMMERSTEIN EQUATIONS.

Roughly speaking, any elliptic semilinear problem, which possesses a Green function can be converted into a nonlinear integral equation of Hammerstein type.

For example, let Ω be a bounded domain in $\mathbb{R}^N$, with a smooth boundary $\partial\Omega$, and let $G(x,y)$ be Green's function related to the Dirichlet problem for the Poisson equation. Then the solution of the linear problem

$$-\Delta v(x) = h(x), \quad x \in \Omega, \quad \text{with} \quad v_{|\partial\Omega} = g(\sigma), \quad \sigma \in \partial\Omega,$$

has the form

$$v(x) = \int_{\Omega} G(x,y)h(y)\, dy + w(x) \quad \text{with} \quad w(x) = -\int_{\partial\Omega} \frac{\partial G(x,\sigma)}{\partial n}\, g(\sigma)\, d\sigma.$$

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Consequently, the solution of the corresponding semilinear problem

(1) $$-\Delta u(x) = f(x,u), \quad x \in \Omega, \quad \text{with} \quad u|_{\partial\Omega} = g(\sigma), \quad \sigma \in \partial\Omega$$

can be represented as

(2) $$u(x) = \int_\Omega G(x,y)\, f(x,u(x))\, dy + w(x).$$

To the same integral equation can be also reduced the Neumann problem for semilinear Poisson equations ([5]). The nonlinear equation (2) is usually called a Hammerstein integral equation. In the integral form (2), the existence of a (weak) solution for the problem (1) is easy to investigate.

We introduce now the linear integral operator

(3) $$Av = \int_\Omega G(\cdot,y)v(y)\, dy$$

and the superposition (nonlinear) operator

$$\mathcal{F}v = f(\cdot,v(\cdot)),$$

to rewrite the Hammerstein equation (2) in operator terms as

$$v - A\mathcal{F}v = w.$$

Replace $u = v - w$ and $Fu = \mathcal{F}(u + w)$; we may assume, without loss of generality, that $w = 0$ and

(4) $$u = AFu\,.$$

Let X be a real function space of possible solutions. Then $AF\colon X \to X$ and the solvability of Hammerstein equations (4) can be studied by using the fixed point theory. This is beyond our discussion.

2. MONOTONE MAPPINGS (basic definitions).

Suppose further that X is a Banach space endowed with the norm $\|\cdot\|$. The set X^* of all continuous linear functionals defined on X is also a Banach space, the dual space of X, with respect to the norm

$$\| h \|_* = \sup_{x \in X} \frac{|(h,x)|}{\|x\|} \quad \text{for every } h \in X^*,$$

where (h,x) denotes the duality pairing between $h \in X^*$ and $x \in X$.

A possible breaking-up of the product AF into two constituent parts (i.e. $F: X \to X^*$ and $A: X^* \to X$) links Hammerstein equations to the theory of operators of monotone type.

For a mapping $T: X \to X^*$ we define $D(T) = \{x \in X | \ Tx \neq \emptyset\}$ the effective domain and $R(T) = \{Tx | \ x \in D(T)\}$ the range. In order that the product, $AF: X \to X$ makes a sense, we assume henceforth that $R(F) \subseteq D(A) \subseteq X^*$.

An operator $T: X \to X^*$ is said to be monotone if

$$(Tu - Tv, u-v) \geq 0 \quad \text{for all } u,v \in D(T) \subseteq X,$$

and T is maximal monotone if it admits no properly monotone extension in $X \to X^*$. In particular, let $\phi: X \to \mathbb{R} \cup \{+\infty\}$ be a convex lower-semicontinuous function. Assume that ϕ is proper, i.e., not merely the constant function $+\infty$, and denote by $D(\phi) = \{x \in X | \phi(x) < +\infty\}$ the effective domain of ϕ. Its subdifferential

$$\partial\phi(v) = \{h \in X^* | \ \phi(w) - \phi(v) \geq (h,w-v) \ \forall \ w \in X\}$$

is a simple example of maximal monotone operator. The elements $h \in \partial\phi(v)$ are called subgradients of ϕ at v. Moreover, if ϕ is Gateaux differentiable, then $\{\partial\phi(v)\} = \{\nabla\phi(v)\} = \{\phi'(v)\}$ is a potential operator.

3. NEMITSKYI OPERATORS IN $L^p(\Omega)$ WITH $1 < p < \infty$.

Nemitskyi operators point out general assumptions in describing nonlinearities arising in the PDE theory.

Let Ω be a bounded domain (or of σ-finite measure) in $\mathbb{R}^N$ and $m \geq 1$ be a natural number. A function $f: \Omega \times \mathbb{R}^m \to \mathbb{R}^m$ satisfies the Caratheodory conditions provided that:

1. $f(\cdot,r)\colon\ \Omega \to \mathbb{R}^m$ is a measurable for each fixed $r \in \mathbb{R}^m$;
2. $f(x,\cdot)\colon\ \mathbb{R}^m \to \mathbb{R}^m$ is continuous for almost all $x \in \Omega$.

With such a function and $u\colon\ \Omega \to \mathbb{R}^m$ we associate the operator $Fu = f(x,u(x))$ called the Nemitskyi or superposition operator. It is easily seen that F carries a measurable function into a measurable function.

PROPOSITION 1. Let $1 < q < \infty$ and p satisfying, in addition, an inequality of the form

$$|f(x,r)| \leq g(x) + c(x) \sum_{j=1}^{m} |r_j|^{\frac{p}{q}}$$

where $g \in L^q(\Omega)$ and $0 < c \in L^\infty(\Omega)$. Then $F\colon\ (L^p(\Omega))^m \to (L^q(\Omega))^m$ is a bounded continuous operator.

In particular, when $q = p'$ with $\frac{1}{p} + \frac{1}{p'} = 1$, then $\frac{p}{q} = p-1$.

PROPOSITION 2. If, furthermore, the function f satisfies the hypothesis

$$\sum_{j=1}^{m} (f_j(x,r) - f_j(x,s))(r_j - s_j) \geq 0 \quad \forall\ r,s \in \mathbb{R}^m,\quad x \in \Omega,$$

then F is a monotone operator in $L^p \times L^{p'}$

For other properties see e.g. [4], Ch. IV.

4. SYMMETRIC POSITIVE LINEAR OPERATORS.

A linear operator $A\colon X^* \to X$ is monotone iff it is positive, i.e.,

$$(Av,v) \geq 0 \quad \forall\ v \in D(A) \subseteq X^*.$$

In the case of linear integral operator (3) with kernel $G \in L^2(\Omega\times\Omega)$ this assumption becomes

$$(Av,v) = \int_\Omega\int_\Omega G(x,y)\ v(x)\ v(y)\ dy \qquad \forall\ v \in L^2(\Omega).$$

When $G(x,y)$ is the above Green function

$$(Av,v) = \int_\Omega v(x)\, dx \int_\Omega G(x,y)\, dy = -\int_\Omega v(x)\, z(x)\, dx,$$

where $z(x) = -\int_\Omega G(x,y)v(y)dy$. Taking into account the Green representation, we derive that $\Delta z = v$ in Ω with $z_{|\partial\Omega} = 0$ and thereby

$$(Av,v) = -\int_\Omega z\, \Delta\, z\, dx = \int_\Omega |\mathrm{grad}\ z|^2 dx \geq 0.$$

We suppose henceforward that $A\colon X^* \to X$ is positive and symmetric, i.e.,

$$(Au,v) = (u,Av)\ \forall\ u,v \in D(A).$$

This last hypothesis follows from the symmetry of Green's function.

FACTORIZATION PRINCIPLE ([1]). Let $A\colon X^* \to X$ be a symmetric monotone linear operator densely defined in X^*. Then there exist a Hilbert space H and a linear map $S\colon X^* \to H$ such that $A = S^*S$, where $S^*\colon H \to X$ is the adjoint map of S. Moreover, S^* is one-to-one and $R(S) \subseteq D(S^*)$.

5. VARIATIONAL METHOD.

The solvability of an operator equation $Tu = 0$ by a variational method means to construct a certain functional whose extreme points serve as roots or pre-images of roots for $Tu = 0$.

THEOREM ([2]). Let X be a reflexive Banach space and X^* its dual space. Let $A\colon X^* \to X$ be a symmetric positive linear operator, densely defined in X^*, and let $\Phi\colon X \to \mathbb{R} \cup \{+\infty\}$ be a subdifferentiable function, which is coercive in the sense that there is a function $c\colon \mathbb{R}_+ \to \mathbb{R}$ bounded from below, such that $c(r) \to \infty$ as $r \to \infty$ and

$$\Phi(v) \geq c(\|v\|) \quad \text{for all} \quad v \in D(\Phi).$$

Then, for every $w \in X$, there exists at least one solution of the equation

$$(I + A\partial\Phi)u = w.$$

Outline of the proof. As above, replacing $\Phi(u)$ by $\Phi(u+w)$, we may assume that $w = 0$. Let H be the Hilbert space of the previous principle with the norm $|||\cdot|||$ and A S^*S. The functional

$$\Psi(u) = \frac{1}{2}|||u|||^2 + \Phi(S^*u) \quad \text{with} \quad D(\Psi) = D(S^*)$$

admits a minimizing sequence $\{u_n\}$ such that $u_n \to u_o$ in H and $S^*u_n \to S^*u_o$ in X are weakly convergent. Then $0 \in \partial\Psi(u_o) = u_o + S\,\partial\Phi(S^*u_o)$. Therefore

$$0 \in S^*u_o + A\,\partial\Phi(S^*u_o),$$

and hence S^*u_o is a solution of the equation $0 \in (I+A\partial\Phi)u$.□

Sometimes in concrete problems, the Banach spaces of possible solutions are not necessarily reflexive, for example, subspaces of $C(\bar{\Omega})$ or $L^1(\Omega)$. In those cases, we can consider a separable space X. The unit ball of its dual X^* is weak* sequentially compact (the topology $\sigma(X^*,X)$). We defined an appropriate subdifferential and extended the above variational procedure to Hammerstein equations on the dual of such a space, ([3]).

6. GENERAL ALGORITHM.

The constructive processes of minimizing sequences corresponding to a convex functional are performed by the solution of numerical systems.

To this end, we impose naturally certain somehow stronger assumptions upon operators and we cofine ourselves to a real Hilbert space H, with the inner product $(\cdot,\cdot)$ and the corresponding norm $\|\cdot\|$. Thus, $A: H \to H$ is a self-adjoint monotone linear operator, which is densely defined in H but not necessarily bounded. Since A is positive, then there exists the square root $A^{\frac{1}{2}}$ also a self-adjoint and monotone operator, with $D(A) \subseteq D(A^{\frac{1}{2}})$. Hence $S = S^* = A^{\frac{1}{2}}$ in this case.

Let $\Phi: H \to \mathbb{R}$ be a Gateaux differentiable functional and $F = \nabla\Phi = \Phi'$.

Assume that Φ is coercive in the sense

$$\Phi(v) \geq \frac{1}{2} a\|v\|^2 + b\|v\|^{\beta} + c , \quad \forall \ v \in H,$$

where $a > 0$, $0 < \beta < 2$ and b,c are arbitrary constants.

As above, we take $w = 0$. Then we are interested in the minimization of the functional

$$\Psi(u) = \frac{1}{2} (Au,u) + \Phi(A^{\frac{1}{2}}u) \quad \text{with} \quad D(\Psi) = D(A^{\frac{1}{2}}).$$

According to the Theorem, if $\Psi(u)$ realizes the infimum at u_o, then $A^{\frac{1}{2}}u_o$ is a solution of the Hammerstein equation $(I+AF)u = 0$. On the other hand, as an extremum of Ψ, u_o is a zero of its Gateaux derivative, i.e.,

$$\nabla\Psi(u,v) = (Au,v) + (\nabla\Phi(A^{\frac{1}{2}}u),A^{\frac{1}{2}}v) = 0 \ , \ \forall \ v \in D(A^{\frac{1}{2}}).$$

Denoting now $(Au,v) = (A^{\frac{1}{2}}u,A^{\frac{1}{2}}v) = a(u,v)$ and $A^{\frac{1}{2}}\Phi A^{\frac{1}{2}} = K$, we rewrite the last equation in the form

$$a(u,v) + \int K(x,u,\nabla u)v = 0 \quad \forall \ v \in D(A^{\frac{1}{2}}).$$

We select a coordinate system $\{w_j\}$ in $D(A^{\frac{1}{2}})$, i.e., a linearly independent system $w_j \in D(A^{\frac{1}{2}})$ dense in the subspace $D(A^{\frac{1}{2}})$. For every natural n, let H_n be the finite-dimensional subspace of $D(A^{\frac{1}{2}})$ spanned by the first n elements. Any generic element in H_n has the form

$$u_n = \sum_{j=1}^{n} r_j w_j$$

where $r_j = r_j(u_n)$ are real unknowns. Substitute these elements in the last equation and get

$$a(w_j,w_i)r_j + G_i(r_1,r_2,\ldots,r_n) = 0 \quad i=1,2,\ldots,n, \tag{5}$$

where

$$G_i(r_1,r_2,\ldots,r_n) = \int K(x, \sum_{j=1}^{n} r_j w_j, \ \nabla(\sum_{j=1}^{n} r_j w_j))w_i, \ i=1,2,\ldots,n.$$

The nonlinear system of numerical equations (5) can be solved, for instance, by the finite elements and the Newton-Kantorovich methods.

Several constructive procedures, in various restrictions upon operators A and F, surveyed in [6] can be regarded as particular realizations of the above algorithm.

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