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ALMOST-PERIODICITY OF COMPACT SOLUTIONS FOR OPERATIONAL-DIFFERENTIAL EQUATIONS. (*)

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Introduction. If the function $u(t)$ from the real line R into a Hilbert space H verifies an equation of the form:

(1) $$u'(t) = A\,u(t)$$

where A is a linear, often unbounded, operator in H (with a domain $D(A) \subset H$), it happens some time that precompactness of the trajectory of $u(t)$ in H is equivalent with its almost-periodicity (for instance, the first result of this kind, by S. Bochner in [1] (see also the fundamental paper of Bochner and von Neumann [2] as well as references [4], [5]). Recently in [3] a special kind of linear operator was considered, which is not included in the class examined in [2]; because of the convexity properties of the solutions of the corresponding equation (1), it turns out that all bounded solutions over R lie on a sphere in H-that is to say, the relation $\|u(t)\| = \|u(0)\|$, $\forall\, t \in R$ holds. Hence, almost-periodicity = normality of precompact solutions follows as usual (see [5]). In this short note we first indicate a simple general principle which includes the case tackled in [3] and then apply it to the equation

(2) $$u'(t) = (\alpha + i\beta)\,B\,u(t)$$

where α, β are real numbers and B is a symmetric operator, of domain $D(B) \subset H$.

1. We shall prove the following:

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<u>Theorem</u>. Consider a Banach space X and then a linear operator A, $D(A) \subset X \to X$. Assume that any regular solution $u(t)$, $R \to D(A)$ of the equation $u'(t) = A\,u(t)$ which is bounded over R verifies the relation $\|u(t)\| = \|u(0)\|$, $\forall\, t \in R$. Then, any solution $v(t)$, $R \to D(A)$ of the same equation which has precompact range in X is almost-periodic.

<u>Proof</u>. Consider any sequence $\{\alpha'_n\}$ of real numbers and extract a subsequence $\{\alpha_n\}$ such that $\{v(\alpha_n)\}_1^\infty$ is a convergent sequence in X. We shall see that the sequence of translated functions $\{v(t + \alpha_n)\}_1^\infty$ is uniformly convergent over the real line. In fact, let us note the equality

(3) $$\frac{d}{dt}[v(t + \alpha_n) - v(t + \alpha_m)] = A[v(t + \alpha_n) - v(t + \alpha_m)]$$

Furthermore, we see the estimates: $\|v(t + \alpha_n) - v(t + \alpha_m)\| \leq 2 \sup_{t\in R}\|v(t)\|$ for all $n,m \in N$; accordingly, using the hypothesis, we get equality:

(4) $$\|v(t + \alpha_n) - v(t + \alpha_m)\| = \|v(\alpha_n) - v(\alpha_m)\|, \quad \forall\, t \in R$$

and this implies the desired uniform convergence.

We shall apply this result in a Hilbert space H, taking $A = \lambda B$ where λ is a complex number while B is an hermitian operator in H with domain $D(B)$. We shall give the (known)

<u>Lemma</u>. If $w'(t) = \lambda B\, w(t)$, $t \in R$, the scalar-valued function $t \to \|w(t)\|^2$ is a convex twice differentiable function.

In fact, we see that $\frac{d}{dt}\langle w(t),w(t)\rangle = (2\mathrm{Re}\lambda)\langle Bw(t),w(t)\rangle$. Use now, say, [6] - page 56 and deduce existence of the second derivative $\frac{d^2}{dt^2}\|w(t)\|^2$ and the equality: $\frac{d^2}{dt^2}\|w(t)\|^2 = 4(\mathrm{Re}\lambda)^2\|B\,w(t)\|^2 \geq 0$. If we assume now that $w(t)$ is a bounded function over R, it follows that the scalar valued function $t \to \|w(t)\|^2$

is constant over R: thus $\|w(t)\|^2 = \|w(0)\|^2$ and accordingly, the solutions $w(t)$ with precompact range are almost-periodic.

Remark. The above considerations extend to solutions of differential-operator equations which are only defined on the non-negative real line. Precisely, in the same way as was done before, we can establish the following:

Proposition. Consider a Banach space X and a linear operator A, $D(A) \subset X \to X$. Assume that any regular solution $u(t)$, $[0,\infty) \to D(A)$ of the equation $u' = Au$ which is bounded over $[0,\infty)$ lies on a sphere in X. Then, any solution $v(t)$ of the same equation with precompact range, belongs to the class $\mathcal{B}(R^+;X)$. (the definition of the class $\mathcal{B}(R^+;X)$ is given in [5] - page 44) As an application of the last result we consider in the Hilbert space H the anti-symmetric operator $A = iB$ where B is hermitian, of domain $D(B) \subset H$. If $w(t)$ is any solution of the equation $w' = iBw$ on the half-line $[0,\infty)$ we find that $\frac{d}{dt}\langle w,w\rangle = 0$, hence $\|w(t)\| = \|w(0)\|$ $\forall t \geq 0$.

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