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HARETU'S CONTRIBUTION TO THE N-BODY PROBLEM

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1. The quantitative period and Haretu.

The history of mechanics has seen two turns from qualitative investigations to quantitative ones and back to a qualitative point of view enriched by topological and differential geometrical ideas. The first period of qualitative work is by far the longest and culminates in the work of Kepler and Galileo. The quantitative period starts with the invention of calculus by Newton and Leibniz and spans 221 years (1687-1889).

During both of these periods, celestial mechanics plays a central role. It was formulated in the quantitative period as the N-body problem and it consists in the following question. Given are N bodies of masses $m_1, m_2, \ldots, m_N$ which move according to Newton's gravitational law. The object is to solve the resulting system of differential equations. The equations of motion are

$$m_i \frac{d^2 \underline{q}_i}{dt^2} = - \frac{\partial U}{\partial \underline{q}_i}, \quad i=1,\ldots,N,$$

where the gravitational potential U is given by

$$U(\underline{q}_1,\ldots,\underline{q}_N) = - \sum_{\substack{i,j=1 \\ i<j}}^{N} \frac{m_i m_j}{\| \underline{q}_i - \underline{q}_j \|} .$$

Newton solved this problem for $N = 2$ and his solution has become standard curriculum in any elementary calculus or physics course. Attempts to solve this problem for $N = 3$ have proved to be the source of some of the most innovative and deep mathematics of the past few hundred years. The question of stability was always one of the main concerns and was analyzed in the quantitative period with series expansion techniques. The works of Laplace (1773), Lagrange (1776), and Poisson (1809) are all connected with attempts to show that successive terms in the series expansion of the axes of the elliptical planetary orbits as functions of the masses are independent of time. Dirichlet made in 1858 a remark to Kronecker that he has found a method of successively approximating the solutions of the N-body problem. Since he died soon afterwards, there was no written proof of this statement. In the same period Weierstrass mentions to S. Kovalevskaia in a letter that he is in the possession of formal series solutions which he cannot show converge. Dirichlet's elusive statement became so famous, that King Oscar of Sweden set a prize for its proof. It is in this context that Haretu makes his mark in 1878 by showing that at the cubic level secular terms do appear. This terminology will be explained in the next section.

Eleven years after Haretu's thesis, Bruns showed that if there was a quantitative method for solving the N-body problem, it had to involve series expansions. Finally, in 1889, Poincaré brings the quantitative period to an end by suggesting in his work that some series expansions might diverge. For his contribution, in spite of not answering Dirichlet's question, Poincaré captures the Swedish King's prize due to the fact that by reintroducing the qualitative point of view, he was able to formulate revolutionary ideas in the study of differential equations with remarkable applications to the N-body problem. The methods founded in Poincaré's work have been in a continuous expansion and most of the modern theory of differential equations is connected in some way or another to Poincaré's ideas. We refer to Moser [4] for a modern account of the N-body problem including the important development of proving the existence of periodic orbits by differential-topological methods (results of Poincaré, Birkhoff, Kolmogorov, Arnold, Moser).

I was prompted to investigate Haretu's work by accidentally discovering a misquotation in the introduction of the famous treatise "Foundations of Mechanics" by Ralph Abraham and Jerrold Marsden [1] . There, Haretu is put on the same list as Laplace, Lagrange, Poisson, and Dirichlet "all of whom claimed to have proved that the solar system was stable". Haretu [2] , infact suggests the opposite: "je fair voir que des termes seculaires apparaissent dans la valeur du demi-grand axe dès la troisieme puissance des masses, ce qui est diamétralement opposé à la conclusion du Memoire de Poisson", "ainsi cette propriété de l'invariabilité des grands axes ... n'existe pas même pour la troisième puissance des masses". Since I was directly involved in the writing of the second edition of this treatise, I feel responsible for not having caught this historical misquotation and take the opportunity here to set the record straight. In fact this mistake was corrected in the 5th printing of the book. I must also confess that I read with great pleasure the first part of Haretu's thesis which employs what we now call the method of reduction of Hamiltonian systems with symmetry, a topic that occupies a central role in my own research in mechanics. The first two sections of Haretu's thesis can be read easily by a modern mathematician with this remark in mind.

2. Haretu's work.

I start by describing the philosophy that permeated the investigations of the constancy of the axes of elliptical planetary orbits during the quantitative period, following Haretu's own introduction to his thesis. In the N-body problem, think of one of the masses as being much greater than all the others, the physical situation being a planetary system with the sun at its center. If all other masses were zero, the points describing the orbits of these fictitious planets would be ellipses with the sun in one of their common foci. Lagrange introduced the method of variation of parameters in order to be able to deal with the real situation of small, but non-zero planetary masses. In this method, one looks at the elliptical elements of the orbit of a planet as being perturbed by the masses of the other planets and finds relatively simple formulas expressing this variation in terms of a perturbative function R. Then, one deals with R by expanding it in a series. By making $R = 0$, i.e. setting all planetary masses equal to zero, one gets elliptical orbits. In first approximation for R, the integration of the resulting equations yields for the elliptic elements perturbation terms of first order with respect to the masses. Now substitute these values just found back into the equations of motion, approximate R quadratically and solve again. Obtain in this way perturbation terms in the elliptical elements that are second order in the masses. Now continue this process to the desired degree of approximation.

In this chain of computations there are terms that could come from summands in R which do not contain time explicitly, but give by integration, terms that depend at least linearly in time. These terms are called secular terms and their effect on the elliptic elements is significant in long time. Their presence raises a serious doubt on the stability of the N-body problem since they hint at significant changes in the shape of the orbits in long time. It has been the purpose of all working on the N-body problem during the quantitative period to show that such terms do not appear. In first approximation, it was shown by Laplace (1773) and then strengthened by Lagrange (1776) who proved that the great axes of the elliptic planetary orbits do not contain secular terms. Poisson (1809) extended this result to quadratic approximations and his computations have been considerably simplified later by Liouville and Puiseux (1841). Tisserand (1876) uses a change of variables employed by Jacobi in his famous paper on the elimination of the nodes to further simplify Poisson's computations for perturbations of second order. This remarkable change of variables represents in fact a very clever reduction procedure in which the canonically conjugate variables on the reduced manifold drop out at once.

Let me explain shortly what Haretu does in modern language. On the cotangent bundle $T^*\mathbb{R}^{3N}$ define the N-body Hamiltonian

$$H = \frac{1}{2}\sum_{i=1}^{N}\frac{1}{m_i}\|\underline{p}^i\|^2 - \sum_{\substack{i,j=1\\ i<j}}^{N}\frac{m_i m_j}{\|\underline{q}_i - \underline{q}_j\|},$$

when $\underline{p}^i = m_i d\underline{q}_i/dt$ and observe that H is invariant under translations, i.e.

$$H(\underline{q}_1+x, \ldots, \underline{q}_N+x, \underline{p}^1, \ldots, \underline{p}^N) = H(\underline{q}_1, \ldots, \underline{q}_N, \underline{p}^1, \ldots, \underline{p}^N).$$

Then the linear momentum $J(\underline{q}_1, \ldots, \underline{q}_N, \underline{p}^1, \ldots, \underline{p}^N) = \underline{p}^1 + \ldots + \underline{p}^N$ is conserved along the flow of the Hamiltonian system determined by H. Under these hypotheses, one can induce a new Hamiltonian system on the quotient manifold $J^{-1}(\mu)/\mathbb{R}^3$ where $\mu \in \mathbb{R}^3$ is arbitrary. This manifold, called the reduced phase space, is symplectic and H induces on it a Hamiltonian system. This is a general phenomenon and the theory for it was developed by Marsden and Weinstein [3] . In fact, it is easy to see that $J^{-1}(\mu)/\mathbb{R}^3$ is symplectically diffeomorphic to $T^*\mathbb{R}^{3(N-1)}$ endowed with the canonical symplectic structure, by invoking the cotangent bundle reduction theorem. Concretely, this reduction procedure boils down in this case to fixing the center of mass of the system and at the same time freezing its linear momentum. What Tisserand and Haretu do, is to carry out this reduction by hand in a very clever way, so that at the end of the process one has the canonical variables on $T^*\mathbb{R}^{3(N-1)}$ explicitly expressed as functions of the old variables and at the same time to write down Lagrange's equations for the new variables. Let me emphasize that, in general, one does not get that much out of a reduction and that, moreover, if one had chosen a different route for carrying out this particular reduction, canonical variables would be far from appearing automatically.

Haretu goes on then and discusses on this reduced manifold the constancy of the great axes of the ellipses to first and second order. This part of his work is very elegant and could be read by a modern day mathematician with ease.

But the most important part of his thesis is the third part, where Haretu using the reduced manifold just described, discusses perturbations of the great axes of the planetary ellipses to third order, starting with Poisson's work already mentioned. This is the bulk of his thesis and consists in a computational "tour de force" resulting in the explicit expressions of secular terms of the form

$$At\cos(\psi+\omega) + Bt^2\cos(\psi+\omega).$$

It is only due to his very careful analysis of all terms involved that he can nail down these expressions, for many of th summands do not involve secular terms. By contradicting thereby Poisson's claim of the constancy of the great axes of the planetary ellipses, Haretu is the first one to openly question the validity of this claim.

Now, over hundred years later, Haretu's work is forgotten mainly due to the fact that it dealt with a very famous problem by means of a method that he himself helped to bury. But at that time, his results created quite a splash and one finds him quoted by many who worked on the N-body problem, including Poincaré.

REFERENCES

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