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AN EXISTENCE THEOREM FOR FUNCTIONAL EQUATIONS OF VOLTERRA TYPE

C. Corduneanu

The aim of this paper is to provide an existence theorem for functional equations of Volterra type in abstract form . Such equations are usually given in the form

$$x(t) = (Vx)(t) \ , \ t \in [0,T) \ , \tag{1}$$

where V stands for a Volterra (causal) operator acting in between some function spaces whose elements are defined on $[0,T)$. We shall assume in this paper that the underlying function space is the space $L^2_{loc}([0,T),R^n)$, and that the operator V takes this space into itself .

The definition we adopt for Volterra (causal) operators can be stated as follows : V will be called a Volterra operator from $L^2_{loc}([0,T),R^n)$ into itself , if from $x(t) = y(t)$ almost everywhere on $[0,\tau]$, there follows $(Vx)(t) = (Vy)(t)$ almost everywhere on the same interval , for every $\tau < T$.

In view of applying the Schauder - Tychonoff fixed point theorem , it will be necessary to secure the continuity of the operator V , as well as its compactness. But these two assumptions will not suffice because one must also construct a convex closed set in the underlying space which is taken by the operator V in some subset of it . At this point , the fact that V is a Volterra operator is of real help . A condition we have indicated in [1] for Caratheodory type existence theorem , in case of ordinary differential equations , can be naturally extended to the case under discussion .

The following theorem provides a set of conditions under which the existence of a global solution is secured for the equation (1) .

Theorem. Consider the functional equation (1) , and assume that V is an operator from $L^2_{loc}([0,T),R^n)$ into itself , verifying the following conditions :

1) V is an operator of Volterra type ;

2) V is continuous in the topology of the space $L^2_{loc}([0,T),R^n)$;

3) V is compact on $L^2_{loc}([0,T),R^n)$;

4) There exists a couple of functions from $[0,T)$ into R_+ , say A(t) and B(t), with A(t) continuous and positive , and B(t) locally integrable , such that $x(t) \in L^2_{loc}([0,T),R^n)$ and

$$\int_0^t \|x(s)\|^2 ds \le A(t) \ , \ t \in [0,T) \ , \tag{2}$$

imply

$$\|(Vx)(t)\|^2 \le B(t) \ , \text{ a.e. on } [0,T), \tag{3}$$

while

$$\int_0^t B(s)ds \le A(t) \ , \ t \in [0,T) \ . \tag{4}$$

Then there exists a solution $x \in L^2_{loc}([0,T),R^n)$ of the equation (1) , such that estimate (2) holds true .

Proof. We choose the space $L^2_{loc}([0,T),R^n)$ as underlying space for the application of the Schauder fixed point theorem . It is known (see , for instance, [3]) that $L^2_{loc}([0,T),R^n)$ is a locally convex complete space .

Let us define now the set S by means of the inequality (2) , i.e.

$$S = \{x \in L^2_{loc}([0,T),R^n) \ ; \ \int_0^t \|x(s)\|^2 ds \le A(t) \ , \ t \in [0,T)\} \ . \tag{5}$$

On behalf of (2) - (4) , it can be easily seen that the operator V takes the set S into itself : $VS \subset S$. On the other hand , the set S is obviously closed and convex in $L^2_{loc}([0,T),R^n)$. For convexity , one has only to notice that the map

$x \to \|x\|^2$ is convex , and then use (2) .

To summarize , we have a set $S \subset L^2_{loc}([0,T),R^n)$ which is closed , convex , and obviously bounded in that space , such that the operator V takes S into a part of S . The Schauder - Tychonoff fixed point theorem applies directly and we obtain the existence of a solution (of course , in S) to equation (1) .

Remark. As far as a local result is concerned with respect to equation (1) , we notice that the existence of the couple A(t) , B(t) , with the properties specified above , excepting (4) , will assure validity of (4) in a right neighborhood of 0 , say $[0,\delta]$, $\delta > 0$.

In order to check how efficient the result given in the theorem is , we will consider the classical Volterra equation

$$x(t) = f(t) + \int_0^t k(t,s,x(s))ds \; , \; t \in [0,T) \; , \tag{6}$$

under the following assumptions on the data :

a) $f \in L^2_{loc}([0,T),R^n)$;

b) $k : D\times R^n \to R^n$, where $D = \{(t,s) \mid 0 \le s \le t < T\}$, is measurable in (t,s) and continuous in x , and moreover

$$|k(t,s,x)| \le k_0(t,s)|x| \; , \tag{7}$$

with $k_0(t,s) \in L^2_{loc}(D,R)$ nonnegative .

Before we discuss consequences of conditions a) and b) , let us notice the fact that condition (7) on k(t,s,x) implies k(t,s,0) = 0 a.e. This might appear as a restriction on the kernel . Actually , in the right hand side of (6) we can subtract and add the term $\int_0^t k(t,s,0)ds$, thus achieving such condition .

Let us denote now , in view of applying the theorem to the equation (6) , by (Vx)(t) the right hand side of equation (6) . It is then obvious that V is a Volterra operator acting from $L^2_{loc}([0,T),R^n)$ into itself . The continuity and compactness of the operator V on the space $L^2_{loc}([0,T),R^n)$ follow from well known results in the theory of integral operators with square integrable kernels (see, for instance, [7]). It is, perhaps, useful to notice that these properties in $L^2_{loc}([0,T),R^n)$ reduce to the corresponding properties in each $L^2([0,t_1],R^n)$, with $t_1 < T$. Of course, condition (7) is a key condition in this respect.

It remains to construct the functions A(t) and B(t) according to the requirements in condition 4) of the theorem. Let A(t) be an arbitrary continuous function on $[0,T)$, assumed also to be nonnegative. It is then easily seen from (6) that B(t) can be chosen as

$$B(t) = 2\|f(t)\|^2 + 2A(t)\int_0^t k_0^2(t,s)ds . \tag{8}$$

Then condition (3) in the statement of the theorem is verified. Let us denote

$$\lambda(t) = \int_0^t k_0^2(t,s)ds \ , t\in[0,T) . \tag{9}$$

On behalf of our assumption on $k_0(t,s)$, $\lambda(t)$ is a locally integrable function on $[0,T)$. If we now take into account the condition (4), then we are conducted to the following integral inequality for the function A(t):

$$A(t) \geq 2\int_0^t \lambda(s)A(s)ds + 2\int_0^t \|f(s)\|^2 ds . \tag{10}$$

In order to make sure A(t) is positive, we will strengthen somewhat the inequality (10). Namely, we will choose A(t) such that the inequality

$$A(t) \geq 2\int_0^t [\lambda(s)+1]A(s)ds + 2\int_0^t \|f(s)\|^2 ds + 1 , \tag{11}$$

is satisfied on $[0,T)$. From (11) one can see that any solution A(t) is positive.

The integral inequality (11) can be easily solved by reduction to a differential inequality, setting

$$y(t) = \int_0^t [\lambda(s)+1]A(s)ds \ , \ t\in[0,T) , \tag{12}$$

which obviously implies $y(0) = 0$, and for $t \in [0,T)$

$$y'(t) \geq 2[\lambda(t)+1]y(t) + [\lambda(t)+1]\ [2\int_0^t \|f(s)\|^2 ds+1] \ . \tag{13}$$

Actually , (13) is valid only almost everywhere on $[0,T)$, but this still allows to obtain a continuous $A(t)$. Indeed , as a possible solution to (13) one can choose the solution of the corresponding ordinary differential equation that vanishes at the origin , which leads to

$$A(t) = y'(t)[\lambda(t)+1]^{-1} = 2y(t) + 2\int_0^t \|f(s)\|^2 ds + 1 \ , \tag{14}$$

From (14) one sees that $A(t)$ is (absolutely) continuous and positive .

Summarizing the above discussion in regard to the Volterra classical equation (6) , we can conclude that conditions a) and b) formulated above assure the existence of at least one solution that belongs to $L^2_{loc}([0,T),R^n)$. Of course,this solution satisfies the equation (6) for almost all t in $[0,T)$.

Since Volterra functional differential equations of the form

$$\dot{x}(t) = (Vx)(t) \ , \ x(0) = x^o \ , \tag{15}$$

can be transformed (by integrating both sides from 0 to t) into a Volterra functional equation of the form (1) , it is likely that the above theorem can be also applied to obtain existence (local or global) for the problem (15) .

By formal integration of both sides of the equation in (15) one obtains the Volterra functional equation

$$x(t) = \int_0^t (Vx)(s)ds + x^o \ , \tag{16}$$

in which the right hand side is an operator that should satisfy the conditions of the above theorem . Instead of proceeding on this path , we will adopt a slightly different approach , coinciding more or less with that pursued in our preceding paper [2] .

Let us formulate now a set of conditions on the operator V in equation (15), under which the existence of a solution for the associated Cauchy problem has at least one solution.

A_1. V is a continuous operator from the space $C_c([0,T),R^n)$ of continuous maps from $[0,T)$ into R^n, with the topology of uniform convergence on compact subintervals, into the space $L_{loc}([0,T),R^n)$.

A_2. There exists a couple of maps from $[0,T)$ into R, say A(t) and B(t), with A(t) continuous and positive and B(t) locally integrable and nonnegative, such that $x \in C_c([0,T),R^n)$ and

$$\|x(t)\| \le A(t)\ ,\quad t \in [0,T)\ , \tag{17}$$

imply

$$\|(Vx)(t)\| \le B(t)\ ,\quad \text{a.e. on } [0,T)\ , \tag{18}$$

while

$$A(t) - A(0) \ge \int_0^t B(s)ds\ ,\quad t \in [0,T)\ . \tag{19}$$

The result concerning problem (15) can be stated as follows :

Let V be a Volterra operator satisfying assumtions A_1 and A_2. Then there exists a solution $x \in AC_{loc}([0,T),R^n)$ of problem (15), satisfying the estimate (16), provided $\|x^o\| \le A(0)$.

Under the above formulated hypotheses, a convenient underlying space for the application of the Schauder fixed point theorem is the space $C_c([0,T),R^n)$. The (Volterra) operator in the right member of (16) satisfies all conditions requested by this theorem.

As shown in [2], this result can be applied to obtain existence of solutions for the classical Volterra integrodifferential equation

$$\dot{x}(t) = f(t,x(t),\int_0^t k(t,s)x(s)ds)\ , \tag{20}$$

under rather natural assumptions on the data.

The theorem given in this paper , as well as its counterpart for Volterra functional differential equations , can be also applied to obtain existence results for differentail equations with delay . Examples in this regard are given in auhor's book [4].

A recent paper by C.N. Friedman [5] also provides existence results for equations that can be written in the form (1) , without assuming the operator to be of Volterra type . An approximating scheme is deviced and used in order to prove the existence of solutions .

In Tricomi's book [6] , the reader can find more details about the Volterra integral equation (6) , under slightly more stringent assumptions . The growth condition (7) is replaced by a Lipschitz type condition , with $k_o(t,s)$ instead of the constant . The convergence (in L^2) of successive approximations is then established .

As a historical note , let us point out that abstract Volterra functional equations have been dealt with by L.Tonelli (1928) , and N.A.Tychonoff (1938) .

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