LIBERTAS MATHEMATICA , VOL. VI (1986)

A NOTE ON ARCHIMEDEAN RIESZ SPACES AND THEIR EXTENDED ORDER DUALS

Wolfgang Filter

If E is an archimedean Riesz space, then by a theorem of S.J. Bernau there exists a locally compact (even a uniquely determined compact) Stonian space X such that E is isomorphic to an order dense Riesz subspace of the space $C_\infty(X)$ of all extended-real-valued continuous functions f on X for which $\{x \in X \mid |f(x)| = \infty\}$ is nowhere dense. We show that the extended order dual E^ρ of E separates E iff X is hyperstonian.
If E^ρ separates E, then we prove that E is isomorphic to E^ρ as Riesz space iff E is universally complete.
Further we introduce the notion of a hypercomplete Riesz space and show that a universally complete Riesz space is hypercomplete if it is separated by the set of its order continuous linear forms.

Notations and terminology

Let E be a Riesz space.

$E_+ := \{x \in E \mid x \geq 0\}$.

A Riesz subspace F of E is called order dense in E iff for all $x \in E_+$ we have $x = \bigvee^{E}_{\substack{y \in F \\ 0 \leq y \leq x}} y$. If E is archimedean this last condition is equivalent to the requirement that for each $x \in E_+ \setminus \{0\}$ there exists $y \in F$ such that $0 < y \leq x$ [AB,1.14].

E is called laterally complete iff the supremum of each disjoint family in E_+ exists.

We say that E is universally complete iff E is Dedekind complete and laterally complete.

A Riesz space F is called universal completion of E iff F is universally complete and E is an order dense Riesz subspace of F.
E^{π} := $\{\xi \in \mathbb{R}^E \mid \xi$ is an order continuous, order bounded linear form on E$\}$.
If E is archimedean we denote by E^{ρ} the extended order dual of E [LM].
Thus if we set $\Phi := \{F \mid F$ is an order dense solid subspace of E$\}$ we have that $E^{\rho} = \bigcup_{F \in \Phi} F^{\pi}$ is the inductive limit of the spaces F^{π} with respect to the injective maps $F^{\pi} \longrightarrow G^{\pi}$, $\xi \longmapsto \xi|_G$ $(F,G \in \Phi,\ G \subset F)$ where we identify $\xi \in F^{\pi}$ and $\eta \in G^{\pi}$ $(F,G \in \Phi)$ if $\xi|_{F \cap G} = \eta|_{F \cap G}$. E^{ρ} is a universally complete Riesz space [LM,1.5].

If Y is a set and $A \subset Y$, then we denote by 1_A^Y or simply by 1_A the characteristic function of A in Y.

Let X be a Hausdorff space.
$C_{\infty}(X)$:= $\{f \in \bar{\mathbb{R}}^X \mid f$ is continuous, $\{x \in X \mid |f(x)| = \infty\}$ is nowhere dense$\}$.
If X is a completely regular Stonian space, $C_{\infty}(X)$ is a universally complete Riesz space [LZ,47.4].
$C(X)$:= $\{f \in \mathbb{R}^X \mid f$ is continuous$\}$.
$C_c(X)$:= $\{f \in C(X) \mid \operatorname{supp} f$ is compact$\}$.
A normal Radon measure on X is a σ-additive map from the set of the relatively compact Borel sets of X in $\mathbb{R}$ which is interior regular with respect to the compact and with respect to the open sets of X.
$M(X)$:= $\{\mu \mid \mu$ is a normal Radon measure on X$\}$.
If X is locally compact, the map $M(X) \longrightarrow C_c(X)^{\pi}$, $\mu \longmapsto \ell_{\mu}$ is a Riesz isomorphism, where ℓ_{μ} denotes the map $C_c(X) \longrightarrow \mathbb{R}$, $f \longmapsto \int f\, d\mu$.
A locally compact Stonian space X is called hyperstonian iff $\bigcup_{\mu \in M(X)} \operatorname{supp} \mu$ is dense in X iff $C_c(X)^{\pi}$ separates $C_c(X)$.

<u>The results</u>

Throughout this paper E denotes an archimedean Riesz space.
We recall Bernau's representation theorem:

<u>Theorem 1</u> [B, th.6 and rem.7] There exist a compact Stonian space X and an injective Riesz homomorphism $u: E \longrightarrow C_{\infty}(X)$ such that uE is an order dense Riesz subspace of $C_{\infty}(X)$. X is unique up to a canonical homeomorphism. u is surjective iff E is universally complete □

To answer the question when the representation space X is hyperstonian we need the following

Proposition 2 If $\bar{E}$ denotes the Dedekind completion of E, then

a) $\bar{E}^{\pi} \longrightarrow E^{\pi}$, $x' \longmapsto x'|_E$ is a Riesz isomorphism.

b) $\bar{E}^{\pi}$ separates $\bar{E}$ iff E^{π} separates E.

Proof: a) follows from [Fr,17Gb)].

b) "$\Rightarrow$": Let $x \in E$, $x \neq 0$. There exists $x' \in \bar{E}^{\pi}$ with $x'(x) \neq 0$. Then $x'|_E \in E^{\pi}$ and $x'|_E(x) \neq 0$.

"$\Leftarrow$": Let $x \in \bar{E}$, $x > 0$. There exist $y \in E$, $0 < y \leq x$, and $x' \in (E^{\pi})_+$ with $x'(y) > 0$. By a) we can find $z' \in (\bar{E}^{\pi})_+$ with $z'|_E = x'$; hence $0 < x'(y) = z'(y) \leq z'(x)$. The assertion now follows from [Z,88.3] □

Theorem 3 Let X be a locally compact Stonian space and $\phi: E \longrightarrow C_\infty(X)$ an injective Riesz homomorphism such that ϕE is order dense in $C_\infty(X)$. Then the following are equivalent:

a) X is hyperstonian.

b) E^{ρ} separates E.

Proof: We remark that by [LM,2.5] E^{ρ} and $C_\infty(X)^{\rho}$ can be identified.

a$\Rightarrow$b: $C_c(X)$, an order dense solid subspace of $C_\infty(X)$, is separated by $C_c(X)^{\pi}$. By [LM,2.6] $C_\infty(X)^{\rho}$ separates $C_\infty(X)$, hence E^{ρ} separates E.

b$\Rightarrow$a: By [LM,2.6] there exists an order dense solid subspace G of E such that G^{π} separates G. Then H, the Dedekind completion of G, is separated by H^{π} (prop.2). H is isomorphic to the solid subspace $\hat{H}$ of $C_\infty(X)$ generated by ϕG.

Now let $A \neq \emptyset$ be an open-compact subset of X. Since $\hat{H}$ is order dense and solid, there exists a nonempty open-compact subset B of A with $1_B \in \hat{H}$, and since $\hat{H}^{\pi}$ separates $\hat{H}$, we find $x' \in (\hat{H}^{\pi})_+$ such that $x'(1_B) > 0$. For $g \in C(B)$ we denote by $\hat{g}$ the canonical extension of g to X (i.e. $\hat{g}|_{X \setminus B} = 0$), and define $\ell: C(B) \longrightarrow \mathbb{R}$, $g \longmapsto x'(\hat{g})$. Then $\ell \in C(B)^{\pi}$ and $\ell > 0$. Hence there exists a normal Radon measure $\mu > 0$ on B such that for all $f \in C(B)$ we have $\ell(f) = \int f d\mu$. The canonical extension $\hat{\mu}$ of μ to X is a normal Radon measure on X, and we have $\hat{\mu}(A) \geq \hat{\mu}(B) = \ell(1_B^B) = x'(1_B^X) > 0$ □

Theorem 4 Let E^ρ separate E. Then the following are equivalent:

a) E is universally complete.

b) There exists a Riesz isomorphism $\phi: E \longrightarrow E^\rho$ such that - if $\phi': E^{\rho\rho} \longrightarrow E^\rho$ denotes the adjoint map of ϕ - $(\phi')^{-1} \circ \phi$ is identical with the canonical embedding of E in $E^{\rho\rho}$.

c) E and E^ρ are isomorphic as Riesz spaces.

Proof: $a \Rightarrow b$: By th.1 there exists a compact Stonian space X such that the Riesz spaces E and $C_\infty(X)$ can be identified. By th.3 X is hyperstonian. b) now follows from [Fi,cor.3.25].

$b \Rightarrow c$ is trivial.

$c \Rightarrow a$ follows from the fact that E^ρ is universally complete □

E is called perfect in the extended sense iff E is canonically isomorphic to $E^{\rho\rho}$. Hence we get as corollary a theorem of Luxemburg and Masterson:

Corollary 5 [LM,2.3] The following are equivalent:

a) E^ρ separates E, and E is universally complete.

b) E is perfect in the extended sense □

If F is a Riesz space and $x \in F_+$, then $y \in F$ is called component of x iff $y \wedge (x - y) = 0$.

Definition 6 Let F be a Riesz space.

We say that $R \subset F^\pi$ satisfies the hc-condition iff $\bigvee^F_{\iota \in I} x_\iota$ exists for each upward-directed family $(x_\iota)_{\iota \in I}$ from F_+ for which $\sup_{\iota \in I} g(x_\iota) < \infty$ for all $g \in R$.

We call F hypercomplete iff there exist a weak unit e of F^ρ and a family $R \subset F^\pi$ of components of e which satisfies the hc-condition □

Proposition 7 Let F be a hypercomplete Riesz space. Then F is canonically isomorphic to $F^{\pi\pi}$; in particular F is Dedekind complete, and F^π separates F.

Proof: It is obvious that F is Dedekind complete.

To show that F^π separates F, let $x \in F_+$ such that $\xi(x) = 0$ for all $\xi \in F^\pi$. There exists $z := \bigvee^F_{n \in \mathbb{N}} nx$, and since F is archimedean, we get $x = 0$. The assertion now follows from [Z,88.3].

From [Z,110.1] we then get $F = F^{\pi\pi}$ □

Hypercomplete Riesz spaces will be investigated in a separate paper. Here we only want to give a condition that implies hypercompleteness. For this purpose we need the following proposition.

Proposition 8 Let E be perfect in the extended sense. Let $R \subset (E^{\pi})_+$ such that $\bigvee^{E^\rho}_{g \in R} g$ exists and is a weak unit of E^ρ. Then R satisfies the hc-condition.

Proof: Set $e := \bigvee^{E^\rho}_{g \in R} g$, S := the solid subspace of E^ρ generated by R, B := the band of E^ρ generated by S. Then $e \in B = E^\rho$ and S is order dense in E^ρ.

Now let $(x_\iota)_{\iota \in I}$ be an upward-directed family from E_+ such that $\sup_{\iota \in I} g(x_\iota) < \infty$ for all $g \in R$. Then $\sup_{\iota \in I} \xi(x_\iota) < \infty$ for all $\xi \in S_+$, and by [LM,2.7] we conclude that $\bigvee^{E}_{\iota \in I} x_\iota$ exists □

Corollary 9 If E^π separates E and E is perfect in the extended sense, then E is hypercomplete.

Proof: There exists a maximal disjoint system $(g_\iota)_{\iota \in I}$ of $(E^\pi)_+$. To show that $e := \bigvee^{E^\rho}_{\iota \in I} g_\iota$ is a weak unit of E^ρ, let $\xi \in (E^\rho)_+$ with $\xi \wedge e = 0$. Then for all $\eta \in E^\pi$ with $0 \leq \eta \leq \xi$ we get $\eta = 0$. By [LM,3.1] E^π is order dense in E^ρ, hence $\xi = 0$.

The assertion now follows from prop.8 □

Combining cor.9 and cor.5 we get

Corollary 10 If E^π separates E and E is universally complete, then E is hypercomplete □

The converse of cor.10 is in general not true. To see this, let $\mathcal{R}$ be a δ-ring of subsets of a set X and M a band of the set $M(\mathcal{R})$ of all real-valued measures on $\mathcal{R}$. Then M is hypercomplete: Put $i_A : M \longrightarrow \mathbb{R},\ \mu \longmapsto \mu(A)$ for all $A \in \mathcal{R}$, $R := \{i_A \mid A \in \mathcal{R}\}$, and $e := \bigvee^{M^\rho}_{A \in \mathcal{R}} i_A$.

But M is in general not laterally complete, as the following proposition shows.

Proposition 11 If X is a set, $\mathcal{R}$ a δ-ring of subsets of X, and M a band of $M(\mathcal{R})$ consisting only of atomfree measures, then the following are equivalent:

a) M is laterally complete.

b) $M = \{0\}$.

Proof: a $\Rightarrow$ b: Assume the existence of $\mu \in M$, $\mu > 0$. We may assume that there exists $A \in R$ with $\mu(A) = \sum_{n \in \mathbb{N}} \frac{1}{n^2}$. Using Liapounov's theorem, one constructs recursively a disjoint sequence $(A_n)_{n \in \mathbb{N}}$ in R with $A_n \subset A$ and $\mu(A_n) = \frac{1}{n^2}$ for all $n \in \mathbb{N}$. Defining $\mu_n := n 1_{A_n} \cdot \mu$ for all $n \in \mathbb{N}$, we get a disjoint sequence $(\mu_n)_{n \in \mathbb{N}}$ from M_+ which obviously is not bounded in M, contradicting a).

b $\Rightarrow$ a is trivial □

Another example in the same direction is furnished by the space ℓ^1.

REFERENCES

[AB] C.D. Aliprantis, O. Burkinshaw, Locally solid Riesz spaces, Academic Press, New York - San Francisco - London, 1978

[B] S.J. Bernau, Unique representation of archimedean lattice groups and normal archimedean lattice rings, Proc. London Math. Soc. 15 (1965), 599-631

[Fi] W. Filter, Dual spaces of $C_\infty(X)$, To appear.

[Fr] D.H. Fremlin, Topological Riesz spaces and measure theory, Cambridge University Press, London - New York, 1974

[LM] W.A.J. Luxemburg, J.J. Masterson, An extension of the concept of the order dual of a Riesz space, Can. J. Math. 19 (1967), 488-498

[LZ] W.A.J. Luxemburg, A.C. Zaanen, Riesz spaces I, North-Holland Publ. Comp., Amsterdam - London, 1971

[Z] A.C. Zaanen, Riesz spaces II, North-Holland Publ. Comp., Amsterdam - New York - Oxford, 1983