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A COINCIDENCE THEOREM IN LINEAR NORMED SPACES

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Abstract. A coincidence theorem of the first author is improved in the general context of the linear normed spaces, using a generalized Mann iterative process and the concept of weak commutativity of the mappings under discussion. Suitable examples are also given.

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1. Introduction

Let X be a Banach space, F a selfmapping of a closed convex subset K of X and I the identity of X. If F is nonexpansive, i.e. $||Fx-Fy|| \leq ||x-y||$ for all $x,y \in K$, Krasnoselskii [11] proved that the sequence of iterates $\{F^n x_o\}$, starting from a given point $x_o \in K$, does not converge necessarily to a fixed point of F, whereas the sequence $\{F_\lambda^n x_o\}$, where

(1) $$F_\lambda = (1 - \lambda) \cdot I + \lambda \cdot F, \ 0 < \lambda \leq 1 ,$$

may converge to a fixed point of F, as shown by Krasnoselskii [11] which assumed

$\lambda = 1/2$, K compact and X uniformly convex. This result was extended by Schaefer [15] considering a general λ.

The scheme (1) has been extended by means of the so-called "Mann iterative process" [12] associated with F and described in the following way: let $x_o \in X$ and $\{x_n\}$ be a sequence defined by

(2) $$x_{n+1} = (1-c_n)\cdot x_n + c_n \cdot Fx_n$$

for n=0,1,2,..., where $\{c_n\}$ is such that

(i) $0 < c_n \leq 1$ for any n=0,1,2,...,

(ii) $\Sigma\, c_n = \infty$.

The scheme (2) has been studied by many authors: cfr. Bose and Mukherjee [1], Das, Singh and Watson [2], Dotson Jr. and Senter [3], Emmanuele [4,5] , Ishikava [7,8] , Massa [13], Rhoades [14].

In this paper we present yet another extension of the Mann iterative process in order to obtain a coincidence theorem in linear normed spaces, using the concept of weak commutativity due to the third author [16]. Illustrative examples are also provided.

2. Preliminaries

We need some basic preliminaries before of presenting our result. Let X be a linear normed space, F and G two selfmappings of X such that G is linear and

(3) $$F(X) \subseteq G(X)$$

Let $x_o \in X$ a given point and y_o a point of X such that $Fx_o = Gy_o$. This can be done since (3) holds. Now, let x_1 be a point of X defined by

$$x_1 = (1-c_o)\cdot x_o + c_o\cdot y_o.$$

Since G is linear, we have

$$Gx_1 = (1-c_o)\cdot Gx_o + c_o\cdot Gy_o = (1-c_o)\cdot Gx_o + c_o\cdot Fx_o$$

and since (3) holds, let y_1 be a point of X such that

$$Fx_1 = Gy_1.$$

Recursively, then we can define two sequences $\{x_n\}$ and $\{y_n\}$ such that

$$x_{n+1} = (1-c_n)\cdot x_n + c_n\cdot y_n,\ Fx_n = Gy_n$$

and

(4) $$Gx_{n+1} = (1-c_n)\cdot Gx_n + c_n\cdot Fx_n$$

for any n=0,1,2... . We denote with $M(Gx_1, c_n, F)$ the sequence defined by (4).

Note that if G=I, we obtain the scheme (2). If $c_n=1$ for every n=0,1,2,..., we deduce the well known iterative process of Jungck [9]:

(5) $$Fx_n = Gx_{n+1}$$

for every n=0,1,2,..., starting from x_o.

As in Rhoades [14], we assume the following requirement:

(iii) $\lim_{n\to\infty} c_n > 0$

instead of (ii).

We shall use also the following definition [16]:

Definition. Let F and G be two selfmappings of a linear normed space X. We say that the pair $\{F,G\}$ is weakly commuting if

$$||FGx - GFx|| \leq ||Fx - Gx||$$

for all $x \in X$.

We note that a commuting pair is clearly weakly commuting but the converse is not true as it is evident in the following Example.

Example 1. Let $X = [0,1]$ with the usual norm. Define $Fx = x/(9+x)$ and $Gx = x/3$ for all $x \in X$. We have

$$||FGx - GFx|| = \frac{x}{27+x} - \frac{x}{27+3x} = \frac{2x^2}{(27+x)\cdot(27+3x)} \leq$$

$$\frac{x^2+6x}{27x+3x} = \frac{x}{3} - \frac{x}{9+x} = ||Fx - Gx||$$

for all $x \in X$, but clearly $FG \neq GF$.

3. A coincidence theorem

Now we are in position to prove the following result:

THEOREM. Let F and G be two selfmappings of a linear normed space X such that $\{F,G\}$ is a weakly commuting pair, G is continuous and linear and condition (3) holds. Further, we have

$$(6) \qquad ||Fx - Fy|| \leq a\cdot||Gx - Gy|| + b\cdot\{||Gx - Fx|| + ||Gy - Fy||\} + c\{||Gx - Fy|| + ||Gy - Fx||\}$$

for all $x,y \in X$, where $a+b+c \geq 0$, $a+2b+2c \leq 1$. Let $x_o \in X$ and $\{c_n\}$ be a sequence satisfying conditions (i) and (iii) such that $M(Gx_1,c_n,F)$ converges to a point u. Then u is a coincidence point of F and G. If b>0, then Fu is the unique common fixed point of F and G.

Proof. Define a sequence $\{F_n\}$ of selfmappings of X by setting

$$F_n(x) = (1-c_n)\cdot Gx + c_n\cdot Fx$$

for all $x\in X$ and for every n=0,1,2... .

Then

$$F_n(x_n) = Gx_{n+1}$$

and

(7) $$\|F_n x - Gx\| = c_n\cdot\|Fx - Gx\|$$

for all $x\in X$ and n=0,1,2... . Since G is linear, we deduce by (7):

$$\|F_n Gx - GF_n x\| = \|(1-c_n)\cdot G^2x + c_n\cdot FGx - (1-c_n)\cdot G^2x - c_n\cdot GFx\| =$$

$$c_n\cdot\|FGx - FGx\| \leq c_n\cdot\|Fx - Gx\| = \|F_n x - Gx\|$$

for all $x\in X$ and n=0,1,2... . So $\{F_n,G\}$ is a weakly commuting pair for any n=0,1,2... .

Using (6), we have

$$\|Fx - Fy\| \leq a\cdot\|Gx - Gy\| + b\cdot\{\|Gx - Fx\| + \|Gy - Fy\|\}$$

$$+ c\cdot\{\|Gx - Gy\| + \|Gy - Fy\| + \|Gy - Gx\| + \|Gx - Fx\|\} =$$

$$(a+2c)\cdot\|Gx-Gy\| + (b+c)\cdot\{\|Gx-Fx\| + \|Gy-Fy\|\} \leq$$

$$\|Gx-Gy\| + \frac{1}{2}\cdot\{\|Gx-Fx\| + \|Gy-Fy\|\}$$

for all $x,y\in X$. Then we deduce

$$\|F_n x-F_n y\| \leq (1-c_n)\cdot\|Gx-Gy\| + c_n\cdot\|Fx-Fy\| \leq$$

$$(1-c_n)\cdot\|Gx-Gy\| + c_n\cdot\|Gx-Gy\| \quad +$$

$$\frac{1}{2}\,c_n\cdot\left\{\|Gx-Fx\| + \|Gy-Fy\|\right\} \quad =$$

$$\|Gx-Gy\| + \frac{1}{2}\left\{\|Gx-F_n x\| + \|Gy-F_n y\|\right\}$$

for all $x,y \in X$ and $n=0,1,2\ldots$. It follows that

$$\|G^2x_{n+1} - F_n u\| \leq \|GF_n x_n - F_n Gx_n\| + \|F_n Gx_n - F_n u\| \leq$$

$$\|Gx_n - F_n x_n\| + \|G^2 x_n - Gu\| \quad +$$

$$\frac{1}{2}\cdot\left\{\|G^2x_n - F_n Gx_n\| + \|Gu - F_n u\|\right\} \leq$$

$$\|Gx_n - Gx_{n+1}\| + \|G^2x_n - Gu\| +$$

$$\frac{1}{2}\cdot\left\{\|G^2x_n - GF_n x_n\| + \|GF_n x_n - F_n Gx_n\|\right\} +$$

$$\frac{1}{2}\cdot\left\{\|Gu - G^2x_{n+1}\| + \|G^2x_{n+1} - F_n u\|\right\}$$

for any $n=0,1,2\ldots$. Since

$$\|GF_n x_n - F_n Gx_n\| \leq \|Gx_n - F_n x_n\| = \|Gx_n - Gx_{n+1}\| \quad ,$$

we deduce from the foregoing inequality:

$$\frac{1}{2}\cdot\|G^2x_{n+1} - F_n u\| \leq \frac{3}{2}\cdot\|Gx-Gx_{n+1}\| + \|G^2x_n - Gu\| \quad +$$

$$\frac{1}{2}\cdot\|G^2x_n - G^2x_{n+1}\| + \frac{1}{2}\cdot\|Gu - G^2x_{n+1}\|$$

for any $n=0,1,2\ldots$. Now, by letting $n\to\infty$ and using the continuity of G,

we find that

$$\lim_{n\to\infty} \|Gu - F_n u\| = 0.$$

Since (7) implies

$$\|Fu - Gu\| = \frac{1}{c_n} \cdot \|Gu - F_n u\|$$

for any n=0,1,2,..., we have, by (iii), as $n \to \infty$, that Fu = Gu, i.e. u is a coincidence point of F and G. Using again the weak commutativity of the pair $\{F,G\}$, we have

$$\|FGu - GFu\| \leq \|Fu - Gu\| = 0,$$

giving thereby

$$GGu = GFu = FGu = FFu.$$

Using (6), we obtain

(8) $$\|Fu - F^2u\| \leq a \cdot \|Gu - GFu\| + b \cdot \{\|Gu - Fu\| + \|GFu - F^2u\|\} + c \cdot \{\|Gu - F^2u\| + \|GFu - Fu\|\} \leq (a+2c) \cdot \|Fu-F^2u\| \leq (1-2b) \cdot \|Fu-F^2u\|,$$

which implies, if b > 0, FFu = Fu and therefore GFu=FFu=Fu.

So Fu=w is a common fixed point of F and G. Let z be a second common fixed point of F and G. By (6), we have

$$\|z - w\| = \|Fz - Fw\| \leq a \cdot \|Gz - Gw\| + c \cdot \{\|Gz-Fw\| + \|Gw-Fz\|\} = (a+2c) \cdot \|z-w\| \leq (1-2b) \cdot \|z-w\|,$$

which implies z=w. This concludes the proof.

Remark 1. Although Khan [10] assumes X to be a Banach space, this assumption is not used in our proof. On the other hand, if {F,G} is a commuting pair and if $c_n=1/2$ for any n=1,2,...,our Theorem includes the cited Theorem 3.1 of Khan [10].

Remark 2. The assumption $b > 0$ is used in our proof in order to guarantee the existence and the uniqueness of the common fixed point of F and G. Such an assumption does not appear in the statement of Theorem 3.1 of Khan [10], but it is explicitly claimed as remark (ii) at page 4 of [10].

Remark 3. If $b = 0$ but we have $a + 2c < 1$, it is easily seen, using the inequality (8), that w is also the unique common fixed point of F and G. For $c=0$, $0<a<1$ and $2b=1 - a$, see a result of Fisher and Sessa [6].

Remark 4. The assumption (ii) on the sequence $\{c_n\}$ is not used in our Theorem. Analogous consideration appears in Rhoades [14] and Das, Singh and Watson [2].

The example below shows that Theorem 3.1 of Khan [10] is not applicable because F does not commute with G whereas our result holds.

Example 2. Let X,F and G as in the Example 1. Note that G is continuous and linear and

$$F(X) = [0,1/10] \subseteq [0,1/3] = G(X).$$

Starting from an arbitrary $x_o \in X$, consider the iterative process (5), i.e. $c_n=1$ for any n=0,1,2.... An easy routine calculation proves that the sequence $M(Gx_1, 1,F)$ consists of the following points of X:

$$\left\{ \frac{x_o}{x_o+9}, \frac{3x_o}{12x_o+81}, \frac{9x_o}{117x_o+729}, \ldots, \frac{3^{n-1}\cdot x_o}{a_n\cdot x_o + 9n}, \ldots \right\},$$

where $\{a_n\}$ is a sequence defined inductively, assuming $a_o=0$, by

$$a_n = 9a_{n-1}+3^{n-1}$$

for any n=1,2,... and since $x_o \leq 1$, it is immediately seen that

$$0 \leq Gx_n = \frac{3^{n-1}\ x_o}{a_n x_o+9^n} \leq \frac{3^{n-1}}{9^n} = \frac{1}{3^n+1}$$

for any n=1,2,... . This implies that $M(Gx_1,1,F)$ converges to 0.

Further, we have

$$\|Fx - Fy\| = \left| \frac{x}{x+9} - \frac{y}{y+9} \right| = \frac{9\cdot|x - y|}{(x+9)\cdot(y+9)} \leq$$

$$\frac{9\cdot|x - y|}{81+|x - y|} \leq \frac{1}{9}\cdot|x - y| = \frac{1}{3}\cdot\|Gx-Gy\|$$

for all $x,y \in X$ and thus all the conditions of our Theorem hold assuming $a=1/3$, $b=c=0$. Note that zero is the unique common fixed point of F and G since

$$a + 2c = 1/3 < 1,$$

as pointed out in Remark 3.

We observe that the weak commutativity in our Theorem is a necessary condition, as is shown in the following

Example 3. Let $X = [0,1]$ with the usual norm. Let $Fx=1/2$ and $Gx=x/2$ for any $x \in X$ and $\{c_n\}$ be a sequence satisfying conditions (i) and (iii). Starting from $x_o=1$, it is easily seen that $x_n=y_n=1$ for any $n=0,1,2...$.

Then

$$M(G1,c_n,F) = \left\{ \frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2}, \ldots \right\}.$$

All the conditions of our Theorem are trivially satisfied except the weak commutativity of F and G since we have

$$\| FGx-GFx \| = \frac{1}{4} > \frac{1-x}{2} = \| Fx - Gx \|$$

for any $x \in (1/2,1]$, but 1/2 is not a coincidence point of F and G.

We also observe that the assumption "Let $x_o \in X$ and $\{c_n\}$ be a sequence such that $M(Gx_1,c_n,F)$ converges to a point u" is necessary as is shown in the following

Example 4. Let $X = [0,+\infty)$ with the usual norm and define $Fx=2x+1$ and $Gx=2x$ for any $x \in X$. Note that G is continuous and linear and

$$F(X) = [1,+\infty) \subseteq [0,+\infty) = G(X).$$

Further, we have

$$\| FGx - GFx \| = 1 = \| Fx - Gx \|$$

for any $x \in X$. Thus F weakly commutes with G and since

$$\| Fx - Fy \| = 2 \cdot \| x - y \| = \| Gx - Gy \| ,$$

the inequality (6) holds with $a=1$, $b=c=0$. Thus all the conditions of our Theorem are satisfied except the aforenamed assumption. Indeed, let $x_o \in X$ arbitrary and $\{c_n\}$ be an arbitrary sequence satisfying conditions (i) and (iii). It is easily seen that $M(Gx_1,c_n,F)$ is defined by

$$Gx_{n+1} = 2x_o + S_n ,$$

where

$$S_n = c_o + c_1 \ldots + c_n$$

for any n=0,1,2... . Since (iii) holds, the sequence $\{S_n\}$ diverges and then $M(Gx_1, c_n, F)$ diverges. On the other hand, F and G do not have coincidence points.

REFERENCES

1. R.K.Bose and R.N.Mukherjee, Approximating fixed points of some mappings, Proc.Amer.Math.Soc. 82 (1981), 603-606.

2. K.M.Das, S.P.Singh and B.Watson, A note on Mann iteration for quasi-nonexpansive mappings, Nonlinear Analysis 6 (1981), 675-676.

3. W.G.Dotson Jr. and H.F.Senter, Approximating fixed points of nonexpansive mappings, Proc.Amer.Math.Soc. 44 (1974), 375-379.

4. G.Emmanuele, Convergence of the Mann-Ishikawa iterative process for nonexpansive mappings, Nonlinear Analysis 6 (1982), 1135-1141.

5. G.Emmanuele, A remark on my paper: "Convergence of the Mann-Ishikawa iterative process for nonexpansive mappings", Nonlinear Analysis 7 (1983), 473-474.

6. B.Fisher and S.Sessa, On a fixed point theorem of Greguš, Internat.J.Math. & Math.Sci., to appear.

7. S.Ishikawa, Fixed points by a new iteration method, Proc.Amer.Math.Soc. 44 (1974), 147-150.

8. S.Ishikawa, Fixed points and iteration of a nonexpansive mapping in a Banach space, Proc.Amer.Math.Soc. 59 (1976), 65-71.

9. G.Jungck, Commuting maps and fixed points, Amer.Math.Monthly 83 (1976), 261-263.

10. M.S.Khan, Fixed points and their approximation in Banach spaces for certain commuting mappings, Glasgow Math.J. 23 (1982), 1-6.

11. M.A.Krasnoselskii, Two remarks about the method of successive approximation, Uspehi Mat. Nauk 63 (1955), 123-127.

12. W.R.Mann, Mean value methods in iteration, Proc.Amer.Math.Soc. 4 (1953), 506-510.

13. S.Massa, Fixed point approximation for quasi-nonexpansive mappings, Le Matematiche 37 (1982), 3-7.

14. B.E.Rhoades, Extensions of some fixed point theorems of Ćirić, Maiti and Pal, Math.Sem.Notes 6 (1978), 41-46.

15. H.Schaefer, Uber die Methode sukzessiver Approximationen, Iber.Deutsch. Math.Verein. 59 (1957), 131-140.

16. S.Sessa, On a weak commutativity condition in fixed point considerations, Publ.Inst.Math. 46 (32) (1982), 149-153.