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# Parametric nonlinear nonhomogeneous singular problems with an indefinite perturbation

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**Abstract**: We consider a singular Dirichlet problem driven by a nonlinear nonhomogeneous differential operator and with a reaction involving the competing effects of a parametric singular term and of an indefinite superlinear Carathéodory perturbation. Using variational tools together with truncation and comparison techniques, we prove an existence and multiplicity result which is global in the parameter $\lambda > 0$ (a bifurcation-type theorem).

**Keywords**: Indefinite perturbation, nonlinear regularity theory, nonlinear maximum principle, minimization of energy functionals, positive solutions.

**MSC2010**: 35J20, 35J75.

*Dedicated to Professor Nicolae Dinculeanu, in memoriam,on the 100th anniversary of his birth.*

## 1 Introduction

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with a $C^2$-boundary $\partial\Omega$. In this paper we study the following parametric singular problem

$$\left\{ \begin{array}{c} -diva\left(Du\left(z\right)\right) = \lambda\theta\left(z\right)u\left(z\right)^{-\eta} + f\left(z,u\left(z\right)\right) \text{ in } \Omega \\ u\mid_{\partial\Omega}= 0,\ u > 0,\ \lambda > 0,\ 0 < \eta < 1. \end{array} \right. \qquad (P_\lambda)$$

The map $a : \mathbb{R}^N \rightarrow \mathbb{R}^N$ defining the differential operator, is continuous, strictly monotone (hence maximal monotone too) and satisfies certain other regularity and growth condition listed in hypotheses $(\mathbf{H}_1)$ stated in Section 2. Theses hypotheses on $a(\cdot)$ provide a general framework in which we can fit many differential operators of interest such the $p-$Laplacian and the $(p, q)-$ Laplacian (the sum of a $p-$Laplacian and of a $q-$Laplacian, $1 < q < p$).

In the reaction, we have the competing effects of a parametric singular term $\lambda\theta\left(z\right)u\left(z\right)^{-\eta}$ ($\lambda > 0$ is a parameter and $\theta \in L^{\infty}\left(\Omega\right)$ such that $\theta\left(z\right) > 0$ for a.a.

$z \in \Omega$) and of a Carathéodory perturbation $f(z,u)$, with $f(z,\cdot)$ superlinear but without satisfying the usual in such cases Ambrosetti-Rabinowitz condition (the AR-condition for short). The interesting feature of $f(z,u)$, is that in general it changes sign (indefinite perturbation).

In most works on singular elliptic problems, the perturbation of the singular term is positive. Such an assumption simplifies the analysis of the problem because then the unique solution of the purely singular problem, is a lower solution of the equation and can be used to bypass the singularity with $C^1-$ functionals.

The main difficulty we encounter when dealing with singular problems, is that the corresponding energy functional is not $C^1$ and so we cannot use on it the minimax methods and results of the critical point theory. When the perturbation is positive, the unique positive solution of the purely singular problem, can be used to rectify this. However, when $f(z,u)$ is sign-changing (indefinite), the above method fails and we need to come up with a new approach to produce positive solutions. As we already mentioned earlier, in the past most "singular" works, assumed that the perturbation is positive. We mention the papers of Haitao [7], Sun-Wu-Long [21] (semilinear equations), Byun-Ko [2], Saoudi-Ghanmi [20], Papageorgiou-Radulescu-Zhang [15], Papageorgiou-Winkert [17] (anisotropic equations) and Giacomoni-Kumar-Sreenadh [6], Papageorgiou-Radulescu-Repovs [14] (nonlinear, nonhomogeneous equations).

Recently Papageorgiou-Zhang [18], considered singular $(p,q)-$ equations with an indefinite perturbation which exhibits an oscillatory behavior near the origin. Using different methods, they proved a theorem on the existence and multiplicity of positive solutions, which is global in $\lambda > 0$.

We mention also the work of Bai-Papageorgiou-Zeng [1], in which the parameter multiplies the indefinite perturbation and the existence and the multiplicity result proved is only local in $\lambda > 0$.

Here using variational tools (involving the direct method of the Calculus of Variations), together with truncations and comparison techniques, we prove an existence and multiplicity theorem which is global in the parameter $\lambda > 0$ (a bifurcation type result). The fact that our result is global in $\lambda > 0$ is the interesting feature of our present work.

## 2 Mathematical Background-Hypotheses

We will be using primarily the Sobolev space $W_0^{1,p}(\Omega)$ $(1 < p < \infty)$ and the Banach space

$$C_0^1\left(\overline{\Omega}\right) = \left\{u \in C^1\left(\overline{\Omega}\right) : u\mid_{\partial\Omega} = 0\right\}.$$

On account of the Poincaré inequality, on $W_0^{1,p}(\Omega)$ we can use the norm

$$\|u\| = \|Du\|_p \text{ for all } u \in W_0^{1,p}(\Omega).$$

where $\|.\|_p$ stands for the $L^p$-norm. The space $C_0^1\left(\overline{\Omega}\right)$ is an ordered Banach space with positive (order) cone

$$C_+ = \left\{u \in C_0^1\left(\overline{\Omega}\right) : u(z) \geq 0 \text{ for all } z \in \overline{\Omega}\right\}.$$

This cone has a nonempty interior given by

$$int\, C_+ = \left\{u \in C_+ : u(z) > 0 \text{ for all } z \in \Omega, \frac{\partial u}{\partial n} |_{\partial\Omega} \leq 0\right\},$$

with $\frac{\partial u}{\partial n} = (Du, n)_{\mathbb{R}^N}$ where $n(\cdot)$ is the outward unit normal on $\partial\Omega$ and $(\cdot,\cdot)_{\mathbb{R}^N}$ denotes the inner product on $\mathbb{R}^N$.

Let $\gamma \in C^1(0,\infty)$ with $\gamma(t) > 0$ for all $t > 0$ be such that

$$0 \leq c \leq \frac{\gamma'(t)\,t}{\gamma(t)} \leq \widehat{c} \text{ and } C_0 t^{p-1} \leq \gamma(t) \leq C_1\left[t^{s-1} + t^{p-1}\right]$$

for all $t > 0$ with $C_0$, $C_1 > 0$, $1 < s < p$.

In what follows let $|.|$ denote the norm in $\mathbb{R}^N$ or $\mathbb{R}^{N^2}$, as well as the absolute value in $\mathbb{R}$. Then our hypotheses on the map $a : \mathbb{R}^N \to \mathbb{R}^N$ are the following:

$(\mathbf{H}_1)$: $a(y) = a_0(|y|)\,y$ for all $y \in \mathbb{R}^N$ with $a_0(t) > 0$ for all $t > 0$ and:

- $(i)$ $a_0 \in C^1(0,\infty)$, $t \to a_0(t)\,t$ is strictly increasing on $(0,\infty)$, $a_0(t)\,t \to 0^+$ as $t \to 0^+$ and
$$\lim_{t\to 0^+} \frac{a_0'(t)\,t}{a_0(t)} > -1;$$
- $(ii)$ $|\nabla a(y)| \leq C_2 \frac{\gamma(|y|)}{|y|}$ for all $y \in \mathbb{R}^N \setminus \{0\}$, some $C_2 > 0$;
- $(iii)$ $\frac{\gamma(|y|)}{|y|} |\xi|^2 \leq (\nabla a(y)\,\xi, \xi)_{\mathbb{R}^N}$ for all $y \in \mathbb{R}^N \setminus \{0\}$, all $\xi \in \mathbb{R}^N$;
- $(iv)$ if $G_0(t) = \int_0^t a_0(s)\,s ds$, then
$$0 \leq pG_0(t) - a_0(t)\,t^2 \text{ for all } t \geq 0.$$

Hypotheses $(\mathbf{H}_1)\,(i)$, $(ii)$, $(iii)$ are dictated by the nonlinear regularity theory of Lieberman [10] and the nonlinear maximum principle of Pucci-Serrin [19] (pp. 111, 120). Hypothesis $(\mathbf{H}_1)\,(iv)$ is a mild condition, serving the needs of our problem. It is satisfied in all cases of interest (see the examples below).

Note that the primitive $G_0(\cdot)$ is strictly convex and strictly increasing on $\mathbb{R}_+$.

We set $G(y) = G_0(|y|)$ for all $y \in \mathbb{R}^N$. Then $G \in C^1\left(\mathbb{R}^N\right)$ and it is strictly convex. Moreover, we have

$$\nabla G(0) = 0,\ \nabla G(y) = G_0'(|y|)\frac{y}{|y|} = a_0(|y|)\, y = a(y) \text{ for all } y \in \mathbb{R}^N \setminus \{0\}.$$

Therefore $G$ is the primitive of $a(\cdot)$. Also from the properties of convex functions, we have

$$G(y) \leq (a(y), y)_{\mathbb{R}^N} \text{ for all } y \in \mathbb{R}^N. \tag{2.1}$$

Using hypotheses $(\mathbf{H}_1)$, we can easily derive the following properties of the map $a(\cdot)$ (see Papageorgiou - Radulescu [12]).

**Lemma 2.1.** *:*

*(a) $a(\cdot)$ is continuous and strictly monotone, thus maximal monotone;*

*(b) $|a(y)| \leq C_3\left(|y|^{s-1} + |y|^{p-1}\right)$ for all $y \in \mathbb{R}^N$, some $C_3 > 0$;*

*(c) $(a(y), y)_{\mathbb{R}^N} \geq \frac{C_0}{p-1}|y|^p$ for all $y \in \mathbb{R}^N$.*

Using this Lemma and (2.1), we deduce the following bilateral growth condition for the primitive $G(\cdot)$ :

**Corollary 2.2.** *If hypotheses $(\mathbf{H}_1)$ hold, then*

$$\frac{C_0}{p(p-1)}|y|^p \leq G(y) \leq C_4(1 + |y|^p) \text{ for all } y \in \mathbb{R}^N, \text{ some } C_4 > 0.$$

**Examples:**

$(a)$ $a(y) = |y|^{p-2}y$ with $1 < p < \infty$.

This map corresponds to the $p-$ Laplacian

$$\triangle_p u = div\left(|Du|^{p-2} Du\right).$$

$(b)$ $a(y) = |y|^{p-2}y + |y|^{q-2}y$ with $1 < q < p < \infty$.

This map corresponds to the $(p,q) -$ Laplacian

$$\triangle_p u + \triangle_q u.$$

$(c)$ $a(y) = \left(1 + |y|^2\right)^{\frac{p-2}{2}} y$ with $1 < p < \infty$.

This map corresponds to the generalized $p-$mean curvature operator

$$div\left(1 + |Du|^2\right)^{\frac{p-2}{2}} Du.$$

$(d)$ $a(y)=|y|^{p-2}y\left(1+\frac{1}{1+|y|^p}\right)$ with $1<p<\infty$.

This map corresponds to the differential operator

$$\triangle_p u+div\left(\frac{|Du|^{p-2}Du}{1+|Du|^p}\right).$$

which appears in plasticity theory (see Fuchs-Seregin [4]).

Consider the operator $V : W_0^{1,p}(\Omega)\to W_0^{1,p}(\Omega)^*=W^{-1,p'}(\Omega)$ $(\frac{1}{p}+\frac{1}{p'}=1)$ defined by

$$\langle V(u),h\rangle =_\Omega (a(Du),Dh)_{\mathbb{R}^N}\,dz \text{ for all } u,\,h\in W_0^{1,p}(\Omega).$$

From Gasinski-Papageorgiou [5] (p.279), we know that this map has the following properties.

**Proposition 2.3.** *If hypotheses* $(\mathbf{H}_1)$ *hold, then the map* $V(\cdot)$ *is bounded (that is, maps bounded sets to bounded sets), continuous, strictly monotone (hence maximal monotone too), coercive and of type* $(S)_+$, *that is, for every sequence* $\{u_n\}_{n\geq 1}\subseteq W_0^{1,p}(\Omega)$ *such that* $u_n\xrightarrow{w}u$ *in* $W_0^{1,p}(\Omega)$ *and*

$$\limsup_{n\to\infty}\langle V(u_n),u_n-u\rangle\leq 0,$$

*one has*

$$u_n\to u \text{ in } W_0^{1,p}(\Omega) \text{ as } n\to\infty.$$

Here $\xrightarrow{w}$ denotes the weak convergence in $W_0^{1,p}(\Omega)$ and $\langle\cdot,\cdot\rangle$ denotes the duality brackets for the pair $\left(W_0^{1,p}(\Omega)^*,W_0^{1,p}(\Omega)\right)$.

If $u,\,v:\Omega\to\mathbb{R}$ are measurable functions with $v(z)\leq u(z)$ for a.a. $z\in\Omega$, then we define

$$[v,u]=\left\{h\in W_0^{1,p}(\Omega):v(z)\leq h(z)\leq u(z)\ \text{for a.a. } z\in\Omega\right\},$$
$$[v)=\left\{h\in W_0^{1,p}(\Omega):v(z)\leq h(z)\ \text{for a.a. } z\in\Omega\right\}$$
$$int_{C_0^1(\overline{\Omega})}[v,u]=\text{interior in } C_0^1\left(\overline{\Omega}\right) \text{ of } [v,u]\cap C_0^1\left(\overline{\Omega}\right).$$

Also, if $u:\Omega\to\mathbb{R}$ is measurable, then we define

$$u^+=\max\{u,0\} \text{ and } u^-=\max\{-u,0\}.$$

Both are measurable $\mathbb{R}_+-$valued functions and

$$u=u^+-u^-,\ |u|=u^++u^-.$$

Moreover, if $u \in W_0^{1,p}(\Omega)$, then $u^{\pm} \in W_0^{1,p}(\Omega)$.

If $X$ is a Banach space and $\varphi \in C^1(X)$, then

$$K_\varphi = \left\{u \in X : \varphi'(u) = 0\right\}$$

is the critical set of $\varphi$. We say that $\varphi(\cdot)$ satisfies the *Cerami condition* (*C-condition*, for short) if it has the following property:

"*Every sequence* $\{u_n\}_{n\geq 1} \subseteq X$ *such that* $\{\varphi(u_n)\}_{n\in\mathbb{N}}$ *is bounded in* $\mathbb{R}$ *and*

$$(1 + \|u_n\|)\,\varphi'(u_n) \to 0 \text{ in } X^* \text{as } n \to \infty$$

*admits a strongly convergent subsequence.*"

Finally, if $u : \Omega \to \mathbb{R}$ is a measurable function, then we write

$$0 \prec u$$

if for every $K \subseteq \Omega$ compact we have

$$0 < c_K \leq u(z) \text{ for a.a. } z \in K.$$

Evidently $0 \prec u$ implies $0 < u(z)$ for a.a. $z \in \Omega$. Moreover, if $u \in C(\Omega)$, $u(z) > 0$ for all $z \in \Omega$, then $0 \prec u$.

The hypotheses on the other data of $(P_\lambda)$ are the following:

$(\mathbf{H}_2)$: $\theta \in L^\infty(\Omega)$, $0 \prec \theta$.

$(\mathbf{H}_3)$: $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that $f(z,0) = 0$ for a.a. $z \in \Omega$ and

$(i)$ $|f(z,x)| \leq \widehat{a}(z)\left(1 + x^{r-1}\right)$ for a.a. $z \in \Omega$, all $x \geq 0$, with $\widehat{a} \in L^\infty(\Omega)$, $p < r < p^*$ where

$$p^* = \begin{cases} \frac{Np}{N-p} & \text{if} \quad p < N \\ +\infty & \text{if} \quad N \leq p; \end{cases}$$

$(ii)$ if $F(z,x) = \int_0^x f(z,s)\,ds$, then $\lim_{x\to\infty} \frac{F(z,x)}{x^p} = +\infty$ uniformly for a.a. $z \in \Omega$;

$(iii)$ there exists $\widehat{\mu} \in \left((r-p)\max\left\{\frac{N}{p}, 1\right\}, p^*\right)$ such that

$$0 < \beta_0 \leq \liminf_{x\to\infty} \frac{f(z,x)\,x - pF(z,x)}{x^{\widehat{\mu}}} \text{ uniformly for a.a.} z \in \Omega;$$

$(iv)$ there exists $\delta \in (0,1)$ such that

$$-\theta(z) \leq f(z,x) \leq 0 \text{ for a.a. } z \in \Omega, \text{ all } 0 \leq x \leq \delta$$

and for every $\rho > 0$, there exists $\widehat{\xi}_\rho > 0$ such that for a.a. $z \in \Omega$,

$$x \to f(z,x) + \widehat{\xi}_\rho x^{p-1}$$

is nondecreasing on $[0,\rho]$.

Since we look for positive solutions and the above hypotheses concern the positive semiaxis $\mathbb{R}_+ = [0,+\infty)$, without any loss of generality we may assume the $f(z,x) = 0$ for a.a. $z \in \Omega$ and $x \leq 0$.

Hypotheses $(\mathbf{H}_3)(ii)$, $(iii)$ imply that for a.a. $z \in \Omega$, $f(z,\cdot)$ is $(p-1)-$ superlinear, but needs not satisfy the AR-condition, which is common when dealing with superlinear elliptic equations (see Willem [22], p.46). Hypothesis $(\mathbf{H}_3)(iv)$ concerns the structure of $f(z,\cdot)$ near zero and we point out that it does not involve an asymptotic condition on $\frac{f(z,x)}{x^{q-1}}$ for some $1 < q \leq p$.

The following function satisfies Hypotheses $(\mathbf{H}_3)$ above but fails to satisfy the AR-condition:

$$f(z,x) = \begin{cases} \theta(z)(x^s - x^\tau) & \text{if } \ 0 \leq x \leq 1 \\ x^{p-1}\ln x & \text{if } \ 1 < x, \end{cases}$$

with $1 < \tau < s$ and $\theta(\cdot)$ as in hypotheses $(\mathbf{H}_3)$.

As we already mentioned in the Introduction, for singular elliptic equations, the energy functional is not $C^1$ and this prevents us from using the minimax results of the critical point theory.

In the next section, we consider some auxiliary elliptic problems, the solutions of which will help us bypass the singularity and deal with $C^1-$ functionals.

# 3 Auxiliary problems

First we consider the following elliptic problem

$$\begin{cases} -diva(Du(z)) = \varepsilon\lambda\theta(z) \ \text{ in } \Omega \\ u\mid_{\partial\Omega} = 0, \ u > 0, \ \varepsilon, \ \lambda > 0. \end{cases} \tag{$Q_\varepsilon$}$$

**Proposition 3.1.** *If hypotheses* $(\mathbf{H}_2)$ *hold and* $\lambda > 0$ *is fixed, then for every* $\varepsilon > 0$ *problem* $(Q_\varepsilon)$ *has a unique positive solution* $\underline{u}_\varepsilon \in int\, C_+$ *and* $\underline{u}_\varepsilon \to 0$ *in* $C_0^1(\overline{\Omega})$ *as* $\varepsilon \to 0^+$.

*Proof.* From Proposition 2.3 we know that $V : W_0^{1,p}(\Omega) \rightarrow W^{-1,p'}(\Omega)$ is maximal monotone, coercive, so it is surjective (see Papageorgiou-Radulescu-Repovs[13], p.135). So, we can find $\underline{u}_\varepsilon \in W_0^{1,p}(\Omega) \setminus \{0\}$ such that

$$V(\underline{u}_\varepsilon) = \varepsilon\lambda\theta(\cdot) \text{ in } W^{-1,p'}(\Omega). \tag{3.1}$$

and on account of the strict monotonicity of $V(\cdot)$, this solution is unique.

Acting on (3.1) with $-\underline{u}_\varepsilon^- \in W_0^{1,p}(\Omega)$, we obtain

$$\frac{C_0}{p-1}\left\|D\underline{u}_\varepsilon^-\right\|_p^p \leq 0 \text{ (see Lemma 2.1)}$$

hence

$$\underline{u}_\varepsilon \geq 0, \ \underline{u}_\varepsilon \neq 0.$$

From Ladyzhenskaya-Uraltseva ([9], p.286), it follows that $\underline{u}_\varepsilon \in L^\infty(\Omega)$. Then using the nonlinear regularity theory of Lieberman [10] we infer that $\underline{u}_\varepsilon \in C_+ \setminus \{0\}$. Note that

$$diva(D\underline{u}_\varepsilon) \leq 0 \text{ in } \Omega.$$

So, from Pucci-Serin ([19], p.120), we conclude that

$$\underline{u}_\varepsilon \in int\ C_+.$$

Using the weak comparison principle (see Pucci-Serrin [19], p.61) we see that $\{\underline{u}_\varepsilon\}_{\varepsilon\in(0,1]} \subseteq int\ C_+$ is nondecreasing and bounded in $W_0^{1,p}(\Omega)$. But then $\{\underline{u}_\varepsilon\}_{\varepsilon\in(0,1]} \subseteq L^\infty(\Omega)$ is bounded and so, from Lieberman [10] we have that $\underline{u}_\varepsilon \in C^{1,\tau}\left(\overline{\Omega}\right)$

$$\begin{aligned} \underline{u}_\varepsilon &\in C^{1,\tau}\left(\overline{\Omega}\right), \ 0<\tau<1 \text{ and } \|\underline{u}_\varepsilon\|_{C^{1,\tau}(\overline{\Omega})} \leq C_5 \\ &\text{for some } C_5>0, \text{ all } 0<\varepsilon\leq 1. \end{aligned} \tag{3.2}$$

Since $\underline{u}_\varepsilon \rightarrow 0$ in $W_0^{1,p}(\Omega)$ as $\varepsilon \rightarrow 0^+$, from (3.2) and recalling that $C^{1,\tau}\left(\overline{\Omega}\right) \hookrightarrow C^1\left(\overline{\Omega}\right)$ compactly, we have

$$\underline{u}_\varepsilon \rightarrow 0 \text{ in } C_0^1\left(\overline{\Omega}\right) \text{ as } \varepsilon \rightarrow 0^+.$$

□

Now choose $\rho > 0$ large enough so that

$$\overline{\Omega} \subseteq B_\rho = \left\{z \in \mathbb{R}^N : |z| < \rho\right\}.$$

On $B_\rho$ we consider the following nonlinear Dirichlet problem

$$\left\{\begin{array}{l} -diva(Du(z)) = \beta \text{ in } B_\rho \\ u\mid_{\partial B_\rho} = 0, \ u>0, \ \beta>0. \end{array}\right. \tag{$S_\beta$}$$

From Papageorgiou-Radulescu-Repovs [14] (Proposition 10), we have:

**Proposition 3.2.** *If hypotheses* $(\mathbf{H}_1)$ *hold, then problem* $(S_\beta)$ *has a unique solution* $\overline{u}_\beta \in int\ C_+\left(\overline{B}_\rho\right)$, $\beta \to \overline{u}_\beta$ *is strictly increasing and* $\overline{u}_\beta \to 0$ *in* $C_0^1\left(\overline{B}_\rho\right)$ *as* $\beta \to 0^+$.

# 4 Positive solutions

We introduce the following sets

$$\mathcal{L} = \{\lambda > 0 : \text{ problem } (P_\lambda) \text{ has a positive solution}\},$$

and

$$\mathcal{S}_\lambda = \text{the set of positive solutions of } (P_\lambda)\,.$$

First we show the nonemptiness of $\mathcal{L}$ and determine the regularity of the elements in $\mathcal{S}_\lambda$.

**Proposition 4.1.** *If hypotheses* $(\mathbf{H}_1)$, $(\mathbf{H}_2)$ *and* $(\mathbf{H}_3)$ *hold, then* $\mathcal{L} \neq \varnothing$ *and for all* $\lambda \in \mathcal{L}$, $\mathcal{S}_\lambda \subseteq int\ C_+$.

*Proof.* Let $\delta > 0$ be as postulated by hypothesis $(\mathbf{H}_3)\,(iv)$. Then on account of Proposition 3.2, for $\beta > 0$ small we have

$$0 \leq \overline{u}_\beta\,(z) \leq \delta \text{ for all } z \in \overline{\Omega}. \tag{4.1}$$

We fix such a $\beta > 0$ and let $m_\beta = \min\limits_{\overline{\Omega}} \overline{u}_\beta > 0$ (recall that $\overline{u}_\beta \in int\ C_+\left(\overline{B}_\rho\right)$ and $\overline{\Omega} \subseteq B_\rho$). Then choose $\lambda > 0$ small such that

$$\lambda\,\|\theta\|_\infty\, m_\beta^{-\eta} \leq \beta. \tag{4.2}$$

With this choice of $\lambda > 0$, by Proposition 3.1, we can find $\varepsilon > 0$ small such that

$$\varepsilon\lambda\,\|\theta\|_\infty \leq \beta,\ 0 \leq \underline{u}_\varepsilon\,(z) \leq \delta \text{ for all } z \in \overline{\Omega}. \tag{4.3}$$

We have

$$\begin{aligned} & -diva\,(D\underline{u}_\varepsilon) - \lambda\theta\,(z)\,\underline{u}_\varepsilon^{-\eta} - f\,(z, \underline{u}_\varepsilon) && (4.4)\\ & = \varepsilon\lambda\theta\,(z) - \lambda\theta\,(z)\,\underline{u}_\varepsilon^{-\eta} - f\,(z, \underline{u}_\varepsilon) \\ & \leq \theta\,(z)\left[\varepsilon\lambda + 1 - \lambda\underline{u}_\varepsilon^{-\eta}\right] \text{ (see hypothesis } (\mathbf{H}_3)\,(iv) \text{ and } (4.3)\,) \\ & \leq 0 \text{ in } \Omega \text{ (by choosing } \varepsilon > 0 \text{ even smaller if necessary).} \end{aligned}$$

Also for $\overline{u}_\beta \in int\ C_+$ we have

$$\begin{aligned} &-diva\left(D\overline{u}_\beta\right) - \lambda\theta\left(z\right)\overline{u}_\beta^{-\eta} - f\left(z, \overline{u}_\beta\right) \qquad (4.5)\\ &\geq \beta - \lambda\left\|\theta\right\|_\infty m_\beta^{-\eta} \text{ (see (4.1) and hypothesis } (\mathbf{H}_3)(iv)\text{)}\\ &\geq 0 \text{ in } \Omega \text{ (see (3.2))}. \end{aligned}$$

Now for $\underline{u}_\varepsilon$ and $\overline{u}_\beta$, we have

$$-diva\left(D\underline{u}_\varepsilon\right) = \varepsilon\lambda\theta\left(z\right) \leq \beta = -diva\left(D\overline{u}_\beta\right) \text{ (see (4.3))}$$

hence

$$\underline{u}_\varepsilon \leq \overline{u}_\beta \text{ (by the weak comparison principle, see [15], p. 61).} \qquad (4.6)$$

Then (4.6) allows the following truncation of the reaction of problem $(P_\lambda)$ (with $\lambda > 0$ small)

$$\widehat{k}_\lambda\left(z, x\right) = \begin{cases} \lambda\theta\left(z\right)\underline{u}_\varepsilon\left(z\right)^{-\eta} + f\left(z, \underline{u}_\varepsilon\left(z\right)\right) & \text{if } x < \underline{u}_\varepsilon\left(z\right)\\ \lambda\theta\left(z\right)x^{-\eta} + f\left(z, x\right) & \text{if } \underline{u}_\varepsilon\left(z\right) \leq x \leq \overline{u}_\beta\left(z\right)\\ \lambda\theta\left(z\right)\overline{u}_\beta\left(z\right)^{-\eta} + f\left(z, \overline{u}_\beta\left(z\right)\right) & \text{if } \overline{u}_\beta\left(z\right) < x. \end{cases} \qquad (4.7)$$

Let $\widehat{K}_\lambda\left(z, x\right) = \int\limits_0^x \widehat{k}_\lambda\left(z, s\right)ds$ and consider the functional $\widehat{\psi}_\lambda : W_0^{1,p}\left(\Omega\right) \to \mathbb{R}$ defined by

$$\widehat{\psi}_\lambda\left(u\right) = \int_\Omega G\left(Du\right)dz - \int_\Omega \widehat{K}_\lambda\left(z, u\right)dz \text{ for all } u \in W_0^{1,p}\left(\Omega\right).$$

From Papageorgiou-Smyrlis ([16], Proposition 3), we know that $\widehat{\psi}_\lambda \in C^1\left(W_0^{1,p}\left(\Omega\right)\right)$. Also from (4.7) and Corollary 2.2, it is clear that $\widehat{\psi}_\lambda\left(\cdot\right)$ is coercive. Moreover, using the Sobolev embedding theorem, we see that $\widehat{\psi}_\lambda\left(\cdot\right)$ is sequentially weakly lower semicontinuous. So, by the Weierstrass-Tonelli theorem, we can find $\widehat{u}_\lambda \in W_0^{1,p}\left(\Omega\right)$ such that

$$\widehat{\psi}_\lambda\left(\widehat{u}_\lambda\right) = \inf\left\{\widehat{\psi}_\lambda\left(u\right) : u \in W_0^{1,p}\left(\Omega\right)\right\}.$$

Then

$$\left\langle\widehat{\psi}'_\lambda\left(\widehat{u}_\lambda\right), h\right\rangle = 0 \text{ for all } h \in W_0^{1,p}\left(\Omega\right),$$

hence

$$\left\langle V\left(\widehat{u}_\lambda\right), h\right\rangle = \int_\Omega \widehat{k}_\lambda\left(z, \widehat{u}_\lambda\right)hdz \text{ for all } h \in W_0^{1,p}\left(\Omega\right). \qquad (4.8)$$

In (4.8) first we use the test function $h = (\underline{u}_\varepsilon - \widehat{u}_\lambda)^+ \in W_0^{1,p}(\Omega)$. We have

$$\left\langle V(\widehat{u}_\lambda), (\underline{u}_\varepsilon - \widehat{u}_\lambda)^+ \right\rangle = \int_\Omega \left[\lambda\theta(z)\,\underline{u}_\varepsilon(z)^{-\eta} + f(z, \underline{u}_\varepsilon(z))\right](\underline{u}_\varepsilon - \widehat{u}_\lambda)^+ dz. \tag{4.9}$$

**Claim:** For every $h \in W_0^{1,p}(\Omega)$,

$$\left|\int_\Omega \left[\lambda\theta(z)\,\underline{u}_\varepsilon(z)^{-\eta}\, h dz\right]\right| < \infty.$$

We have

$$\begin{aligned} &\left|\textstyle\int_\Omega \left[\lambda\theta(z)\,\underline{u}_\varepsilon(z)^{-\eta}\, h dz\right]\right| \le \textstyle\int_\Omega \left[\lambda\theta(z)\,\underline{u}_\varepsilon(z)^{-\eta}\,|h|\, dz\right] \\ &\le \textstyle\int_\Omega \underline{u}_\varepsilon(z)^{1-\eta}\,\dfrac{\lambda\theta(z)\,|h|}{\underline{u}_\varepsilon} dz. \end{aligned} \tag{4.10}$$

Recall that $\underline{u}_\varepsilon \in int\, C_+$. So, we can find $C_6 > 0$ such that

$$C_6\widehat{d} \le \underline{u}_\varepsilon \text{ with } \widehat{d}(z) = dist(z, \partial\Omega) \text{ for all } z \in \overline{\Omega}.$$

Using this in (4.10), we have

$$\begin{aligned} &\left|\int_\Omega \left[\lambda\theta(z)\,\underline{u}_\varepsilon(z)^{-\eta}\, h dz\right]\right| \\ \le\ & \lambda C_7 \int_\Omega \frac{|h|}{\widehat{d}} dz \text{ for some } C_7 > 0 \\ \le\ & \lambda C_8 \|h\| \text{ for some } C_8 > 0 \\ & \text{(using Hardy's inequality, see [13], p. 66)} \\ <\ & \infty. \end{aligned}$$

This proves the Claim.

We return to (4.9) and use the Claim. We have

$$\begin{aligned} &\left\langle V(\widehat{u}_\lambda), (\underline{u}_\varepsilon - \widehat{u}_\lambda)^+ \right\rangle \\ \ge\ & \int_\Omega \theta(z)\left[\lambda\underline{u}_\varepsilon(z)^{-\eta} - 1\right](\underline{u}_\varepsilon - \widehat{u}_\lambda)^+ dz \\ & \text{(see (4.3) and hypothesis } (\mathbf{H}_3)(iv)\text{)} \\ \ge\ & \int_\Omega \varepsilon\lambda\theta(z)(\underline{u}_\varepsilon - \widehat{u}_\lambda)^+ \text{ (see (4.4))} \\ =\ & \left\langle V(\underline{u}_\varepsilon), (\underline{u}_\varepsilon - \widehat{u}_\lambda)^+ \right\rangle \text{ (see Proposition 3.1)} \end{aligned}$$

therefore

$$\underline{u}_\varepsilon \le \widehat{u}_\lambda \text{ for } \lambda > 0 \text{ small (see Proposition 2.3).} \tag{4.11}$$

Next in (4.8) we use the test function $h=(\widehat{u}_\lambda-\overline{u}_\beta)^+\in W_0^{1,p}(\Omega)$. Then we have

$$\left\langle V(\widehat{u}_\lambda),(\widehat{u}_\lambda-\overline{u}_\beta)^+\right\rangle=\int_\Omega\left[\lambda\theta(z)\overline{u}_\beta(z)^{-\eta}+f(z,\overline{u}_\beta(z))\right](\widehat{u}_\lambda-\overline{u}_\beta)^+dz \quad (4.12)$$

(see (4.7). Reasoning as in the Claim, we show that

$$\left|\int_\Omega\left[\lambda\theta(z)\overline{u}_\beta(z)^{-\eta}hdz\right]\right|<\infty \text{ for all } h\in W_0^{1,p}(\Omega).$$

Then from (4.12) and (4.5), we have

$$\left\langle V(\widehat{u}_\lambda),(\widehat{u}_\lambda-\overline{u}_\beta)^+\right\rangle\le\left\langle V(\widehat{u}_\beta),(\widehat{u}_\lambda-\overline{u}_\beta)^+\right\rangle$$

hence

$$\widehat{u}_\lambda\le\overline{u}_\beta \text{ (see Proposition 2.3).}$$

So, we have proved that

$$\widehat{u}_\lambda\in[\underline{u}_\varepsilon,\overline{u}_\beta] \text{ for } \lambda>0 \text{ small.} \quad (4.13)$$

Then from (4.13), (4.7) and (4.8) we infer that

$$\widehat{u}_\lambda\in\mathcal{S}_\lambda \text{ for } \lambda>0 \text{ small.}$$

Next let $\lambda\in\mathcal{L}$ and let $u\in\mathcal{S}_\lambda$. Then from Marino-Winkert [11], we have that $u\in L^\infty(\Omega)$. Also, from the previous argument we know that we can find $\varepsilon=\varepsilon(\lambda)>0$ small such that $\underline{u}_\varepsilon\le u$. Since $u\in W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$ using hypothesis $(\mathbf{H}_3)(i)$, we have

$$\begin{aligned}\left|\lambda\theta(z)u^{-\eta}+f(z,u)\right| &\le \lambda\theta(z)u^{-\eta}+C_9 \text{ for some } C_9>0\\ &\le C_{10}u^{-\eta} \text{ for some } C_{10}=C_{10}(\lambda)>0\\ &\le C_{10}\underline{u}_\varepsilon^{-\eta}\\ &\le C_{11}\widehat{d}^{-\eta} \text{ for some } C_{11}>0 \text{ (since } \underline{u}_\varepsilon\in int\ C_+).\end{aligned}$$

Invoking Theorem 1.7 of Giacomoni-Kumar-Sreenadh [6], we conclude that $u\in int\ C_+$. So, $\mathcal{S}_\lambda\subseteq int\ C_+$. □

Next we show that $\mathcal{L}$ is connected.

**Proposition 4.2.** *If hypotheses* $(\mathbf{H}_1)$, $(\mathbf{H}_2)$ *and* $(\mathbf{H}_3)$ *hold,* $\lambda\in\mathcal{L}$ *and* $0<\mu<\lambda$, *then* $\mu\in\mathcal{L}$.

*Proof.* Since $\lambda \in \mathcal{L}$, we can find $u_\lambda \in \mathcal{S}_\lambda \subseteq int\ C_+$ (see Proposition 4.1). So, by Proposition 3.1 for $\varepsilon > 0$ small, we have

$$\underline{u}_\varepsilon(z) \leq u_\lambda(z) \text{ and } 0 \leq \underline{u}_\varepsilon(z) \leq \delta \text{ for all } z \in \overline{\Omega}. \tag{4.14}$$

with $\delta > 0$ as postulated by hypothesis $(\mathbf{H}_3)\,(iv)$. Then we can define the following truncation of the reaction in problem $(P_{\mu:})$ :

$$k_\mu(z,x) = \begin{cases} \mu\theta(z)\,\underline{u}_\varepsilon(z)^{-\eta} + f(z,\underline{u}_\varepsilon(z)) & \text{if } x < \underline{u}_\varepsilon(z) \\ \mu\theta(z)\,x^{-\eta} + f(z,x) & \text{if } \underline{u}_\varepsilon(z) \leq x \leq u_\lambda(z) \\ \mu\theta(z)\,u_\lambda(z)^{-\eta} + f(z,u_\lambda(z)) & \text{if } u_\lambda(z) < x. \end{cases} \tag{4.15}$$

We set $K_\mu(z,x) = \int\limits_0^x k_\mu(z,s)\,ds$ and consider the functional $\psi_\mu : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\psi_\mu(u) = \int_\Omega G(Du)\,dz - \int_\Omega K_\mu(z,u)\,dz \text{ for all } u \in W_0^{1,p}(\Omega).$$

We know that $\psi_\mu \in C^1\left(W_0^{1,p}(\Omega)\right)$ (see [16]). It is clear from (4.15) that $\psi_\mu$ is coercive. Also the Sobolev embedding theorem implies that $\psi_\mu(\cdot)$ is sequentially weakly lower semicontinuous. So, by the Weierstrass-Tonelli theorem, we can find $u_\mu \in W_0^{1,p}(\Omega)$ such that

$$\psi_\mu(u_\mu) = \inf\left\{\psi_\mu(u) : u \in W_0^{1,p}(\Omega)\right\}.$$

Then

$$\left\langle \psi_\mu'(u_\mu), h\right\rangle = 0 \text{ for all } h \in W_0^{1,p}(\Omega),$$

hence

$$\langle V(u_\mu), h\rangle = \int_\Omega k_\mu(z,u_\mu)\,h dz \text{ for all } h \in W_0^{1,p}(\Omega). \tag{4.16}$$

In (4.16) we choose the test function $h = (\underline{u}_\varepsilon - u_\mu)^+ \in W_0^{1,p}(\Omega)$ and obtain

$$\begin{aligned} &\left\langle V(u_\mu), (\underline{u}_\varepsilon - u_\mu)^+\right\rangle \\ &= \textstyle\int_\Omega \left[\mu\theta(z)\,\underline{u}_\varepsilon(z)^{-\eta} + f(z,\underline{u}_\varepsilon(z))\right](\underline{u}_\varepsilon - u_\mu)^+\,dz \text{ (see (4.15))} \\ &\geq \textstyle\int_\Omega \theta(z)\left[\mu\underline{u}_\varepsilon(z)^{-\eta} - 1\right](\underline{u}_\varepsilon - u_\mu)^+\,dz \text{ (see (4.14) and hypothesis } (\mathbf{H}_3)\,(iv)) \\ &\geq \textstyle\int_\Omega \varepsilon\mu\theta(z)\,(\underline{u}_\varepsilon - u_\mu)^+\,dz \text{ (choosing } \varepsilon \in (0,1) \text{ even smaller if necessary)} \\ &= \left\langle V(\underline{u}_\varepsilon), (\underline{u}_\varepsilon - u_\mu)^+\right\rangle \text{ (see Proposition 3.1)),} \end{aligned}$$

therefore

$$\underline{u}_\varepsilon \le u_\mu .$$

Next in (4.16) we use the test function $h = (u_\mu - u_\lambda)^+ \in W_0^{1,p}(\Omega)$. Then

$$\begin{aligned}
&\left\langle V(u_\mu), (u_\mu - u_\lambda)^+ \right\rangle \\
&= \int_\Omega \left[ \mu\theta(z)\, u_\lambda^{-\eta} + f(z, \underline{u}_\varepsilon(z)) \right] (u_\mu - u_\lambda)^+ \, dz \text{ (see (4.15))} \\
&\le \int_\Omega \lambda\theta(z) \left[ u_\lambda^{-\eta} + f(z, u_\lambda) \right] (u_\mu - u_\lambda)^+ \, dz \text{ (since } \mu < \lambda) \\
&= \left\langle V(u_\lambda), (u_\mu - u_\lambda)^+ \right\rangle \text{ (since } u_\lambda \in \mathcal{S}_\lambda)),
\end{aligned}$$

therefore

$$u_\mu \le u_\lambda .$$

So, we have proved that

$$u_\mu \in [\underline{u}_\varepsilon, u_\lambda]. \tag{4.17}$$

From (4.17), (4.15) and (4.16), we conclude that

$$u_\mu \in \mathcal{S}_\mu \subseteq int\ C_+,$$

therefore $\mu \in \mathcal{L}$. □

An interesting by-product of the above proof, is the following corollary establishing a monotonicity- type property of the solution multifunction $\lambda \to \mathcal{S}_\lambda$.

**Corollary 4.3.** *If hypotheses* $(\mathbf{H}_1)$, $(\mathbf{H}_2)$ *and* $(\mathbf{H}_3)$ *hold,* $\lambda \in \mathcal{L}$, $u_\lambda \in \mathcal{S}_\lambda$ *and* $0 < \mu < \lambda$, *then* $\mu \in \mathcal{L}$ *and there exists* $u_\mu \in \mathcal{S}_\mu$ *such that* $u_\mu \le u_\lambda$.

In fact we can improve the conclusion of the above corollary and establish strict monotonicity of the solution multifunction $\lambda \to \mathcal{S}_\lambda$.

**Proposition 4.4.** *If hypotheses* $(\mathbf{H}_1)$, $(\mathbf{H}_2)$ *and* $(\mathbf{H}_3)$ *hold,* $\lambda \in \mathcal{L}$, $u_\lambda \in \mathcal{S}_\lambda$ *and* $0 < \mu < \lambda$, *then* $\mu \in \mathcal{L}$ *and there exists* $u_\mu \in \mathcal{S}_\mu$ *such that* $u_\lambda - u_\mu \in int\ C_+$.

*Proof.* From Corollary 4.3, we already know that $\mu \in \mathcal{L}$ and that there exists $u_\mu \in \mathcal{S}_\mu$ such that

$$0 \le u_\mu \le u_\lambda \tag{4.18}$$

Let $\rho = \|u_\lambda\|_\infty$ (recall $u_\lambda \in int\ C_+$) and let $\widehat{\xi}_\rho > 0$ be as postulated by hypothesis $(\mathbf{H}_3)(iv)$. Then we have

$$\begin{aligned}
&-diva\left(Du_{\mu}\right)+\widehat{\xi}_{\rho}u_{\mu}^{p-1}-\lambda\theta\left(z\right)u_{\mu}^{-\eta} && (4.19)\\
&=f\left(z,u_{\mu}\right)+\widehat{\xi}_{\rho}u_{\mu}^{p-1}-\left(\lambda-\mu\right)\theta\left(z\right)u_{\mu}^{-\eta}\ \text{(since } u_{\mu}\in\mathcal{S}_{\mu}\text{)}\\
&\leq f\left(z,u_{\lambda}\right)+\widehat{\xi}_{\rho}u_{\lambda}^{p-1}\ \text{(see (4.18), } \left(\mathbf{H}_{3}\right)(iv) \text{ and recall } \mu<\lambda\text{)}\\
&=-diva\left(Du_{\lambda}\right)+\widehat{\xi}_{\rho}u_{\lambda}^{p-1}-\lambda\theta\left(z\right)u_{\lambda}^{-\eta}\ \text{(since } u_{\lambda}\in\mathcal{S}_{\lambda}\text{ ).}
\end{aligned}$$

Note that

$$0\prec\left(\lambda-\mu\right)\theta\left(\cdot\right)u_{\mu}^{-\eta}\ \text{(see hypotheses } \left(\mathbf{H}_{2}\right)\text{).}$$

So, by (4.19) and Proposition 7 of Papageorgiou-Radulescu-Repovs [14], we have

$$u_{\lambda}-u_{\mu}\in int\ C_{+}.$$

□

At this point we only know that $\mathcal{L}$ is an interval with left endpoint 0. Next we show that $\mathcal{L}$ is a bounded interval.

So, let

$$\lambda^{*}=\sup\mathcal{L}.$$

**Proposition 4.5.** *If hypotheses* $\left(\mathbf{H}_{1}\right)$, $\left(\mathbf{H}_{2}\right)$ *and* $\left(\mathbf{H}_{3}\right)$ *hold, then* $\lambda^{*}<+\infty$.

*Proof.* Hypotheses $\left(\mathbf{H}_{3}\right)$ imply that we can find $\lambda_{0}>0$ such that

$$\lambda_{0}\theta\left(z\right)x^{-\eta}+f\left(z,x\right)\geq\theta\left(z\right)x^{p-1}\ \text{for a.a. } z\in\Omega,\ \text{all } x\geq0. \tag{4.20}$$

To see this, note that hypotheses $\left(\mathbf{H}_{3}\right)(ii)$, $(iii)$ imply that

$$\lim_{x\to\infty}\frac{f\left(z,x\right)}{x^{p-1}}=+\infty\ \text{uniformly for a.a. } z\in\Omega.\text{for a.a. } z\in\Omega.$$

Indeed note that on account of hypothesis $\left(\mathbf{H}_{3}\right)(iii)$ we have

$$f\left(z,x\right)x-pF\left(z,x\right)\geq0\ \text{for a.a. } z\in\Omega,\ \text{all } x\geq M.$$

This combined with hypothesis $\left(\mathbf{H}_{3}\right)(iii)$ gives the claimed limit.

Therefore we can find $M>1$ such that

$$f\left(z,x\right)\geq\left\Vert\theta\right\Vert_{\infty}x^{p-1}\ \text{for a.a. } z\in\Omega,\ \text{all } x\geq M,$$

hence

$$\lambda\theta\left(z\right)x^{-\eta}+f\left(z,x\right)\geq\theta\left(z\right)x^{p-1}\ \text{for a.a. } z\in\Omega,\ \text{all } x\geq M. \tag{4.21}$$

Hypothesis $(\mathbf{H}_3)\,(iv)$ implies that

$$\lambda\theta\,(z)\,x^{-\eta}+f\,(z,x)\geq\theta\,(z)\left[\lambda x^{-\eta}-1\right]\ \text{ for a.a. } z\in\Omega,\ \text{ all } x\geq M.$$

We can find $\lambda_0'>0$ such that

$$\lambda_0'\delta^{-\eta}-1\geq\delta^{p-1}$$

hence

$$\lambda_0' x^{-\eta}-1\geq x^{p-1}\text{ for all } 0<x\leq\delta\text{ (see Hypothesis } (\mathbf{H}_3)\,(iv)\,),$$

therefore

$$\lambda_0'\theta\,(z)\,x^{-\eta}+f\,(z,x)\geq\theta\,(z)\,x^{p-1}\text{ for a.a. } z\in\Omega,\ \text{ all } 0<x\leq\delta. \tag{4.22}$$

For a.a. $z\in\Omega$, and for $\delta\leq x\leq M$, we have

$$\lambda\theta\,(z)\,x^{-\eta}+f\,(z,x)\geq\theta\,(z)-\|\widehat{a}\|_\infty\left(1+x^{p-1}\right)\ \text{ (see Hypothesis } (\mathbf{H}_3)\,(i)\,),.$$

Then, we can find $\lambda_0''>0$ such that

$$\lambda_0''\theta\,(z)\,M^{-\eta}-\|\widehat{a}\|_\infty\left(1+\delta^{p-1}\right)\geq\|\theta\|_\infty M^{p-1},$$

therefore

$$\lambda_0''\theta\,(z)\,x^{-\eta}+f\,(z,x)\geq\theta\,(z)\ \text{ for a.a. } z\in\Omega,\ \text{ all } \delta\leq x\leq M. \tag{4.23}$$

Finally combining $(4.21)$, $(4.22)$ and $(4.23)$ we see that $(4.20)$ holds with

$$\lambda_0=\max\left\{\lambda_0',\lambda_0''\right\}.$$

Now let $\lambda>\lambda_0$ and suppose that $\lambda\in\mathcal{L}$. Then we can find $u_\lambda\in\mathcal{S}_\lambda\subseteq int\ C_+$. Let $\Omega_0\subseteq\Omega$ be open with $C^2-$boundary and $\overline{\Omega}_0\subseteq\Omega$. We set

$$m_0=\min_{\overline{\Omega}_0}u_\lambda>0.$$

For $\varepsilon\in(0,1)$ small we define $m_0^\varepsilon=m_0+\varepsilon$. Let $\widehat{\xi}_\rho>0$ be as postulated by hypothesis $(\mathbf{H}_3)\,(iv)$. Then in the open set $\Omega_0$ we have

$$\begin{aligned}
&-diva\,(Dm_0^\varepsilon)+\widehat{\xi}_\rho\,(m_0^\varepsilon)^{p-1}-\lambda\theta\,(z)\,(m_0^\varepsilon)^{-\eta}\\
&\leq\widehat{\xi}_\rho m_0^{p-1}-\lambda\theta\,(z)\,m_0^{-\eta}+\chi\,(\varepsilon)\ \text{ with } \chi\,(\varepsilon)\to0\text{ as }\varepsilon\to0^+\\
&\leq\left(\widehat{\xi}_\rho+\theta\,(z)\right)m_0^{p-1}-\lambda\theta\,(z)\,m_0^{\cdot\eta}+\chi\,(\varepsilon)\\
&\leq(\lambda_0-\lambda)\,\theta\,(z)\,m_0^{-\eta}+f\,(z,m_0)+\widehat{\xi}_\rho m_0^{p-1}+\chi\,(\varepsilon)\ \text{ (see 4.20)}\\
&\leq f\,(z,u_\lambda)+\widehat{\xi}_\rho u_\lambda^{p-1}\text{ for }\varepsilon\in(0,1)\ \text{ small}\\
&\qquad\qquad\qquad\text{(since }\lambda>\lambda_0\text{ and }\chi\,(\varepsilon)\to0\text{ as }\varepsilon\to0^+)\\
&=-diva\,(Du_\lambda)+\widehat{\xi}_\rho u_\lambda-\lambda\theta\,(z)\,u_\lambda^{-\eta}\text{ in }\Omega_0.
\end{aligned}$$

Then by Proposition 6 of Papageorgiou-Radulescu-Repovs [14], we have

$$m_0^{\varepsilon} < u_\lambda(z) \text{ for all } z \in \Omega_0, \text{ all } \varepsilon \in (0,1) \text{ small,}$$

a contradiction. This means that $\lambda \notin \mathcal{L}$ and so, we conclude that

$$0 < \lambda^* \leq \lambda_0 < \infty.$$

□

So, we have

$$(0, \lambda^*) \subseteq \mathcal{L} \subseteq (0, \lambda^*].$$

For $\lambda \in (0, \lambda^*)$ we establish the multiplicity of positive solutions.

**Proposition 4.6.** *If hypotheses* $(\mathbf{H}_1)$, $(\mathbf{H}_2)$ *and* $(\mathbf{H}_3)$ *hold and* $0 < \lambda < \lambda^*$, *then problem* $(P_\lambda)$ *has at least two positive solutions* $u_0$ *and* $\widehat{u} \in int\ C_+$.

*Proof.* Let $\gamma \in (\lambda, \lambda^*)$. We know that $\gamma \in \mathcal{L}$ and so, we can find $u_\gamma \in S_\gamma \subseteq int\ C_+$. From Proposition 4.4 we know that for $\varepsilon \in (0,1)$ small we can find $u_0 \in S_\lambda \subseteq int\ C_+$ such that

$$u_0 \in [\underline{u}_\varepsilon, u_\gamma] \text{ with } u_\gamma - u_0 \in int\ C_+ \tag{4.24}$$

With $\rho = \|u_\gamma\|_\infty$ and $\widehat{\xi}_\rho$ as in hypothesis $(\mathbf{H}_3)(iv)$, we have

$$\begin{aligned}
&-diva\,(D\underline{u}_\varepsilon) + \widehat{\xi}_\rho \underline{u}_\varepsilon^{p-1} - \lambda\theta(z)\,\underline{u}_\varepsilon^{-\eta} \\
&\leq \varepsilon\lambda\theta(z) + \widehat{\xi}_\rho \underline{u}_\varepsilon^{p-1} \text{ (see Proposition 3.1)} \\
&\leq f(z, \underline{u}_\varepsilon) + \widehat{\xi}_\rho \underline{u}_\varepsilon^{p-1} \text{ (see (4.3) and hypothesis } (\mathbf{H}_3)(iv)) \\
&\leq f(z, u_0) + \widehat{\xi}_\rho u_0^{p-1} \text{ (see (4.24) and hypothesis } (\mathbf{H}_3)(iv)) \\
&= -diva\,(Du_0) + \widehat{\xi}_\rho u_0^{p-1} - \lambda\theta(z)\,u_0^{-\eta} \text{ (since } u_0 \in \mathcal{S}_\lambda \text{ )}
\end{aligned}$$

therefore $u_0 - \underline{u}_\varepsilon \in int\ C_+$ (see [14], Proposition 7).

We have proved that

$$u_0 \in int_{C_0^1(\overline{\Omega})}\,[\underline{u}_\varepsilon, u_\gamma]. \tag{4.25}$$

As in the proof of Proposition 4.2 let

$$k_\lambda(z,x) = \begin{cases} \lambda\theta(z)\,\underline{u}_\varepsilon(z)^{-\eta} + f(z, \underline{u}_\varepsilon(z)) & \text{if } x < \underline{u}_\varepsilon(z) \\ \lambda\theta(z)\,x^{-\eta} + f(z,x) & \text{if } \underline{u}_\varepsilon(z) \leq x \leq u_\gamma(z). \\ \lambda\theta(z)\,u_\gamma(z)^{-\eta} + f(z, u_\gamma(z)) & \text{if } u_\gamma(z)_\varepsilon(z) < x. \end{cases}$$

Set $K_\lambda(z,x) = \int\limits_0^x k_\lambda(z,s)\,ds$ and consider the $C^1-$functional $\psi_\lambda : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\psi_\lambda(u) = \int_\Omega G(Du)\,dz - \int_\Omega K_\lambda(z,u)\,dz \text{ for all } u \in W_0^{1,p}(\Omega).$$

Also let $k_\lambda^* : \Omega \times \mathbb{R} \to \mathbb{R}$ the Carathéodory function defined by

$$k_\lambda^*(z,x) = \begin{cases} \lambda\theta(z)\,\underline{u}_\varepsilon(z)^{-\eta} + f(z,\underline{u}_\varepsilon(z)) & \text{if } x < \underline{u}_\varepsilon(z) \\ \lambda\theta(z)\,x^{-\eta} + f(z,x) & \text{if } \underline{u}_\varepsilon(z) \le x. \end{cases} \tag{4.26}$$

We set $K_\lambda^*(z,x) = \int\limits_0^x k_\lambda^*(z,s)\,ds$ and consider the $C^1-$functional $\psi_\lambda^* : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\psi_\lambda^*(u) = \int_\Omega G(Du)\,dz - \int_\Omega K_\lambda^*(z,u)\,dz \text{ for all } u \in W_0^{1,p}(\Omega).$$

From (4.26) and the nonlinear regularity theory, we have

$$K_{\psi_\lambda^*} \subseteq [\underline{u}_\varepsilon) \cap int\ C_+ \tag{4.27}$$

It is clear that

$$\psi_\lambda^* \mid_{[\underline{u}_\varepsilon, u_\gamma]} = \psi_\lambda \mid_{[\underline{u}_\varepsilon, u_\gamma]}. \tag{4.28}$$

Recall that $u_0$ is a global minimizer of $\psi_\lambda^*(\cdot)$ (see the proof of Proposition 7). So, from (4.28) and (4.25) it follows that $u_0$ is a local $C_0^1(\overline{\Omega})-$minimizer of $\psi_\lambda^*(\cdot)$, hence also

$$u_0 \text{ is a local } W_0^{1,p}(\overline{\Omega}) - \text{minimizer of } \psi_\lambda^*(\cdot) \tag{4.29}$$

(see Proposition $A4$ of Papageorgiou-Radulescu-Zhang [15]).

From (4.27) we see that we may assume that $K_{\psi_\lambda^*}$ is finite or otherwise we already have an infinity of positive solutions in *int* $C_+$. Then (4.29) and Theorem 5.7.6, p. 449, of [13], imply that we can find $\rho \in (0,1)$ small such that

$$\psi_\lambda^*(u_0) < \inf\{\psi_\lambda^*(u) : \|u - u_0\| = \rho\} = m_\lambda. \tag{4.30}$$

If $u \in int\ C_+$, then on account of hypothesis $(\mathbf{H}_3)(ii)$ and (4.26) we have

$$\psi_\lambda^*(tu) \to -\infty \text{ as } t \to +\infty. \tag{4.31}$$

Moreover, using hypotheses $(\mathbf{H}_1)\,(iv)$ and $(\mathbf{H}_3)\,(iii)$ and reasoning as in the Claim in the proof of Proposition 4 in Papageorgiou-Radulescu-Zhang [15], we show that

$$\psi_\lambda^* \text{ satisfies the } C\text{-condition.} \tag{4.32}$$

Then $(4.30)$, $(4.31)$ and $(4.32)$ permit the use of the mountain pass theorem. So, we can find $\widehat{u} \in K_{\psi_\lambda^*(u)} \subseteq [\underline{u}_\varepsilon) \cap int\ C_+$ such that

$$\psi_\lambda^*\ (u_0) < m_\lambda \leq \psi_\lambda^*\ (\widehat{u})\,.$$

Then $\widehat{u} \in int\ C_+$ is the second positive solution of $(P_\lambda)$ distinct from $u_0$. □

We have to determine the status of the critical parameter $\lambda^* > 0$.

**Proposition 4.7.** *If hypotheses* $(\mathbf{H}_1)$, $(\mathbf{H}_2)$ *and* $(\mathbf{H}_3)$ *hold, then* $\lambda^* \in \mathcal{L}$.

*Proof.* First some observations. For every $\lambda \in \mathcal{L}$ and $\theta \in (\lambda, \lambda^*)$, $u_\theta \in \mathcal{S}_\theta \subseteq int\ C_+$, we have $\mathcal{S}_\lambda \cap [0, u_\theta] \neq \varnothing$. The set $\mathcal{S}_\lambda$ is downward directed (that is, if $u, \widetilde{u} \in \mathcal{S}_\lambda$ then there exists $y \in \mathcal{S}_\lambda$ such that $y \leq u$, $y \leq \widetilde{u}$, see Filippakis-Papageorgiou [3]). Then using Lemma 3.10 of Hu-Papageorgiou [8], p.118, we can find $\{u_n\}_{n\in\mathbb{N}} \subseteq \mathcal{S}_\lambda \cap [0, u_\theta]$ decreasing such that

$$\inf \mathcal{S}_\lambda = \inf_{n\in\mathbb{N}} u_n.$$

Using Theorem 1.7 of Giacomoni-Kumar-Sreenadh [6] we infer that $u_n \in C^{1,\tau}\left(\overline{\Omega}\right)$ for some $\tau \in (0,1)$ and $\{u_n\}_{n\in\mathbb{N}} \subseteq C^{1,\tau}\left(\overline{\Omega}\right)$ is bounded. Exploiting the compact embedding of $C^{1,\tau}\left(\overline{\Omega}\right)$ into $C^1\left(\overline{\Omega}\right)$, we see that we may assume that $u_n \to v_\lambda$ in $C^1\left(\overline{\Omega}\right)$ and $\underline{u}_\varepsilon \leq v_\lambda$ for $\varepsilon \in (0,1)$ small.

So, $v_\lambda \in \mathcal{S}_\lambda$ and $v_\lambda = \inf \mathcal{S}_\lambda$.

Now consider an increasing sequence $\{\lambda_n\}_{n\in\mathbb{N}} \subseteq (0, \lambda^*)$ such that $\lambda_n \to (\lambda^*)^-$ and let $u_n \in \mathcal{S}_{\lambda_n} \subseteq int\ C_+$, $n \in \mathbb{N}$. We have

$$-diva\,(Du_n) = \lambda_n \theta\,(z)\, u_n^{-\eta} + f\,(z, u_n) \geq \lambda_1 \theta\,(z)\, u_n^{-\eta} + f\,(z, u_n) \text{ in } \Omega,\ n \in \mathbb{N}. \tag{4.33}$$

Using (4.33) and a standard truncation technique for problem $(P_{\lambda_1})$ (we truncate the reaction of $(P_{\lambda_1})$ from above at $u_n\,(z)$ and use (4.33)), we produce $u_1^n \in \mathcal{S}_{\lambda_1} \subseteq int\ C_+$ such that

$$0 \leq u_1^n \leq u_n \text{ for all } n \in \mathbb{N},$$

hence

$$v_{\lambda_1} \leq u_n \text{ for all } n \in \mathbb{N}$$

therefore

$$\underline{u}_\varepsilon \leq v_{\lambda_1} \leq u_n \text{ for } \varepsilon \in (0,1) \text{ small, all } n \in \mathbb{N}. \tag{4.34}$$

Let $\psi^*_{\lambda_n}(\cdot)$ be as in the proof of Proposition 4.6 (with $\lambda_n$ instead of $\lambda$). We know that we can have $u_n$ to be a minimizer of $\psi_{\lambda_n}$ with $\gamma > \lambda_n$ and so from (4.28) we infer that

$$\begin{aligned} &\psi^*_{\lambda_n}(u_n) \le \psi^*_{\lambda_n}(\underline{u}_\varepsilon) && (4.35)\\ &\le \int_\Omega (a(D\underline{u}_\varepsilon), D\underline{u}_\varepsilon)_{\mathbb{R}^N}\,dz - \int_\Omega K^*_{\lambda_n}(z, \underline{u}_\varepsilon)\,dz \text{ (see } (\mathbf{H}_1)(iv)) \\ &= \int_\Omega \left(\lambda_n \theta(z)\left(\varepsilon - \underline{u}_\varepsilon^{-\eta}\right) + f(z, \underline{u}_\varepsilon)\right)\underline{u}_\varepsilon dz \text{ (see (4.26))} \\ &\le 0 \text{ for } \varepsilon \in (0,1) \text{ small.} \end{aligned}$$

Since $\left(\psi^*_{\lambda_n}\right)'(u_n) = 0$ for all $n \in \mathbb{N}$ , using (4.35) and reasoning as in the Claim in the proof of Proposition 4 of [15] we obtain (at least for a subsequence) that

$$u_n \to u^* \text{ in } W_0^{1,p}(\Omega)$$

From (4.34) it follows that $u^* \neq 0$ and so $u^* \in \mathcal{S}_{\lambda^*}$, hence $\lambda^* \in \mathcal{L}$. □

So, summarizing, we can state the following existence and multiplicity result for problem $(P_\lambda)$, which is global in $\lambda > 0$. Our result extends Theorem 3.12 of [18].

For $\lambda \in (0, \lambda^*)$ we have multiplicity of positive solutions

**Theorem 4.8.** *If hypotheses* $(\mathbf{H}_1)$, $(\mathbf{H}_2)$ *and* $(\mathbf{H}_3)$ *hold, then there exists* $\lambda^* > 0$ *such that*

$(a)$ *for all* $\lambda \in (0, \lambda^*)$, *problem* $(P_\lambda)$ *has at least two positive solutions* $u_0, \widehat{u} \in$ *int* $C_+$;

$(b)$ *for* $\lambda = \lambda^*$, *problem* $(P_\lambda)$ *has at least one positive solutions* $u^* \in$ *int* $C_+$;

$(c)$ *for* $\lambda > \lambda^*$, *problem* $(P_\lambda)$ *has no positive solutions.*

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