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Noncommutative Perspectives of Operator Monotone Functions in Hilbert Spaces

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Abstract: Assume that the function $f:[0,\infty)\to\mathbb{R}$ is operator monotone in $[0,\infty)$ and has the representation

$$f(t)=f(0)+bt+\int_0^\infty \frac{t\lambda}{t+\lambda}dw(\lambda),$$

where $b\geq 0$ and w is a positive measure on $(0,\infty)$. In this paper we obtained among others that

$$\begin{aligned}&\mathcal{P}_f(B,P)-\mathcal{P}_f(A,P)\\&=b(B-A)+\int_0^\infty \lambda^2\left[\int_0^1 P((1-t)A+tB+\lambda P)^{-1}(B-A)\right.\\&\left.\times((1-t)A+tB+\lambda P)^{-1}Pdt\right]dw(\lambda)\end{aligned}$$

for all A, B, $P>0$. Applications for *weighted operator geometric mean* and *relative operator entropy* are also provided.

Keywords: Operator monotone functions, Noncommutative perspectives, Weighted operator geometric mean, Relative operator entropy.
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1 Introduction

Consider a complex Hilbert space $(H,\langle\cdot,\cdot\rangle)$. An operator T is said to be positive (denoted by $T\geq 0$) if $\langle Tx,x\rangle\geq 0$ for all $x\in H$ and also an operator T is said to be *strictly positive* (denoted by $T>0$) if T is positive and invertible. A real valued continuous function f on $(0,\infty)$ is said to be operator monotone if $f(A)\geq f(B)$ holds for any $A\geq B>0$.

We have the following integral representation for the power function when $t>0$, $r\in(0,1]$, see for instance [1, p. 145]

$$t^{r-1}=\frac{\sin(r\pi)}{\pi}\int_0^\infty \frac{\lambda^{r-1}}{\lambda+t}d\lambda. \tag{1.1}$$

Observe that for $t>0$, $t\neq 1$, we have

$$\int_0^u \frac{d\lambda}{(\lambda+t)(\lambda+1)}=\frac{\ln t}{t-1}+\frac{1}{1-t}\ln\left(\frac{u+t}{u+1}\right)$$

for all $u>0$.

By taking the limit over $u\to\infty$ in this equality, we derive

$$\frac{\ln t}{t-1}=\int_0^\infty \frac{d\lambda}{(\lambda+t)(\lambda+1)},$$

which gives the representation for the logarithm

$$\ln t=(t-1)\int_0^\infty \frac{d\lambda}{(\lambda+1)(\lambda+t)} \tag{1.2}$$

for all $t>0$.

In 1934, K. Löwner [10] had given a definitive characterization of operator monotone functions as follows, see for instance [1, p. 144-145]:

Theorem 1.1. *A function $f:[0,\infty)\to\mathbb{R}$ is operator monotone in $[0,\infty)$ if and only if it has the representation*

$$f(t)=f(0)+bt+\int_0^\infty \frac{t\lambda}{t+\lambda}dw(\lambda) \tag{1.3}$$

where $b\geq 0$ and a positive measure w on $(0,\infty)$ such that

$$\int_0^\infty \frac{\lambda}{1+\lambda}dw(\lambda)<\infty.$$

We recall the important fact proved by Löwner and Heinz that states that the power function $f:[0,\infty)\to\mathbb{R}$, $f(t)=t^\alpha$ is an operator monotone function for any $\alpha\in[0,1]$, [9]. The function ln is also operator monotone on $[0,\infty$.

For other examples of operator monotone functions, see [7] and [8].

Let f be a continuous function defined on the interval I of real numbers, B a selfadjoint operator on the Hilbert space H and A a positive invertible operator on H. Assume that the spectrum $Sp\left(A^{-1/2}BA^{-1/2}\right)\subset \mathring{I}$. Then by using the continuous functional calculus, we can define the *perspective* $\mathcal{P}_f(B,A)$ by setting

$$\mathcal{P}_f(B,A):=A^{1/2}f\left(A^{-1/2}BA^{-1/2}\right)A^{1/2}.$$

If A and B are commutative, then

$$\mathcal{P}_f(B,A)=Af\left(BA^{-1}\right)$$

provided $Sp\left(BA^{-1}\right) \subset \mathring{I}$.

For any function $f:(0,\infty)\to\mathbb{R}$ the transpose $\tilde{f}$ of f is defined by

$$\tilde{f}(x)=xf\left(x^{-1}\right),\ x>0.$$

It is well known that (see for instance [12]), if $f:(0,\infty)\to\mathbb{R}$ is continuous on $(0,\infty)$, then for all A, $B>0$,

$$\mathcal{P}_{\tilde{f}}(A,B)=\mathcal{P}_f(B,A). \tag{1.4}$$

If f is nonnegative and operator monotone on $(0,\infty)$, then $\tilde{f}$ is operator monotone on $(0,\infty)$, see [12].

The following inequality is of interest, see [12]:

Theorem 1.2. *Assume that f is nonnegative and operator monotone on $(0,\infty)$. If $A\geq C>0$ and $B\geq D>0$, then*

$$\mathcal{P}_f(A,B)\geq\mathcal{P}_f(C,D). \tag{1.5}$$

It is well known that (see [3] and [2] or [4]), if f is an *operator convex function* defined in the positive half-line, then the mapping

$$(B,A)\mapsto\mathcal{P}_f(B,A)$$

defined in pairs of positive definite operators, is operator convex.

If $f_\nu:[0,\infty)\to[0,\infty)$, $f_\nu(t)=t^\nu$, $\nu\in[0,1]$, then

$$P_{f_\nu}(B,A):=A^{1/2}\left(A^{-1/2}BA^{-1/2}\right)^\nu A^{1/2}=:A\sharp_\nu B,$$

is the *weighted operator geometric mean* of the positive invertible operators A and B with the weight ν.

We define the *weighted operator arithmetic mean* by

$$A\nabla_\nu B:=(1-\nu)A+\nu B,\ \nu\in[0,1].$$

It is well known that the following *Young's type inequality* holds:

$$A\sharp_\nu B\leq A\nabla_\nu B$$

for any $\nu\in[0,1]$.

If we take the function $f=\ln$, then

$$P_{\ln}(B,A):=A^{1/2}\ln\left(A^{-1/2}BA^{-1/2}\right)A^{1/2}=:S(A|B),$$

is the *relative operator entropy*, for positive invertible operators A and B.

Kamei and Fujii [5], [6] defined the *relative operator entropy* $S(A|B)$, for positive invertible operators A and B, which is a relative version of the operator entropy considered by Nakamura-Umegaki [11].

2 Main Results

We start to the following identity of interest:

Lemma 2.1. *Assume that the function $f : [0,\infty) \to \mathbb{R}$ is operator monotone in $[0,\infty)$ and has the representation (1.3). Then for all U, $V > 0$ we have*

$$\begin{aligned} f(V) - f(U) &= b(V-U) \\ &\quad + \int_0^\infty \lambda^2 \left[\int_0^1 ((1-t)U + tV + \lambda)^{-1} \right. \\ &\quad \left. \times (V-U)((1-t)U + tV + \lambda)^{-1} dt \right] dw(\lambda). \end{aligned} \tag{2.1}$$

Proof. Since the function $f : [0,\infty) \to \mathbb{R}$ is operator monotone in $(0,\infty)$ and has the representation (1.3), then for U, $V > 0$ we have the representation

$$f(V) - f(U) = b(V-U) + \int_0^\infty \lambda \left[V(V+\lambda)^{-1} - U(U+\lambda)^{-1} \right] dw(\lambda). \tag{2.2}$$

Observe that for $\lambda > 0$

$$\begin{aligned} & V(V+\lambda)^{-1} - U(U+\lambda)^{-1} \\ &= (V+\lambda-\lambda)(V+\lambda)^{-1} - (U+\lambda-\lambda)(U+\lambda)^{-1} \\ &= (V+\lambda)(V+\lambda)^{-1} - \lambda(V+\lambda)^{-1} - (U+\lambda)(U+\lambda)^{-1} + \lambda(U+\lambda)^{-1} \\ &= \lambda \left[(U+\lambda)^{-1} - (V+\lambda)^{-1} \right]. \end{aligned}$$

Therefore, (2.2) becomes, see also [8]

$$f(V) - f(U) = b(V-U) + \int_0^\infty \lambda^2 \left[(U+\lambda)^{-1} - (V+\lambda)^{-1} \right] dw(\lambda). \tag{2.3}$$

Let T, $S > 0$. The function $f(t) = -t^{-1}$ is operator monotonic on $(0,\infty)$, operator Gâteaux differentiable and the Gâteaux derivative is given by

$$\nabla f_T(S) := \lim_{t\to 0} \left[\frac{f(T+tS) - f(T)}{t} \right] = T^{-1}ST^{-1} \tag{2.4}$$

for T, $S > 0$.

Consider the continuous function f defined on an interval I for which the corresponding operator function is Gâteaux differentiable and for C, D selfadjoint operators with spectra in I we consider the auxiliary function defined on $[0,1]$ by

$$f_{C,D}(t) = f((1-t)C + tD), \ t \in [0,1].$$

If $f_{C,D}$ is Gâteaux differentiable on the segment $[C,D] := \{(1-t)C + tD,\ t \in [0,1]\}$, then we have, by the properties of the Bochner integral, that

$$f(D) - f(C) = \int_0^1 \frac{d}{dt}(f_{C,D}(t))\,dt = \int_0^1 \nabla f_{(1-t)C+tD}(D-C)\,dt. \tag{2.5}$$

If we write this equality for the function $f(t) = -t^{-1}$ and $C,\ D > 0$, then we get the representation

$$C^{-1} - D^{-1} = \int_0^1 ((1-t)C + tD)^{-1}(D-C)((1-t)C+tD)^{-1}\,dt. \tag{2.6}$$

Now, if we replace in (2.6) $C = U + \lambda$ and $D = V + \lambda$ for $\lambda > 0$, then

$$\begin{aligned}&(U+\lambda)^{-1} - (V+\lambda)^{-1} \\ &= \int_0^1 ((1-t)U + tV + \lambda)^{-1}(V-U)((1-t)U + tV + \lambda)^{-1}\,dt.\end{aligned} \tag{2.7}$$

By the representation (2.3), we derive (2.1). □

Theorem 2.2. *Assume that the function $f : [0,\infty) \to \mathbb{R}$ is operator monotone in $[0,\infty)$ and has the representation (1.3). Then for all $A,\ B,\ P > 0$ we have*

$$\begin{aligned}&\mathcal{P}_f(B,P) - \mathcal{P}_f(A,P) \\ &= b(B-A) + \int_0^\infty \lambda^2 \left[\int_0^1 P((1-t)A + tB + \lambda P)^{-1}(B-A)\right. \\ &\left.\times ((1-t)A + tB + \lambda P)^{-1} P dt\right] dw(\lambda).\end{aligned} \tag{2.8}$$

Proof. If we take $V = P^{-1/2}BP^{-1/2}$ and $U = P^{-1/2}AP^{-1/2}$ in (2.1), then we get

$$\begin{aligned}&f\left(P^{-1/2}BP^{-1/2}\right) - f\left(P^{-1/2}AP^{-1/2}\right) \\ &= b\left(P^{-1/2}BP^{-1/2} - P^{-1/2}AP^{-1/2}\right) \\ &+ \int_0^\infty \lambda^2 \left[\int_0^1 \left((1-t)P^{-1/2}AP^{-1/2} + tP^{-1/2}BP^{-1/2} + \lambda\right)^{-1}\right. \\ &\times \left(P^{-1/2}BP^{-1/2} - P^{-1/2}AP^{-1/2}\right) \\ &\left.\times \left((1-t)P^{-1/2}AP^{-1/2} + tP^{-1/2}BP^{-1/2} + \lambda\right)^{-1} dt\right] dw(\lambda).\end{aligned} \tag{2.9}$$

Observe that

$$P^{-1/2}BP^{-1/2}-P^{-1/2}AP^{-1/2}=P^{-1/2}\left(B-A\right)P^{-1/2},$$

and

$$\left(1-t\right)P^{-1/2}AP^{-1/2}+tP^{-1/2}BP^{-1/2}+\lambda=P^{-1/2}\left(\left(1-t\right)A+tB+\lambda P\right)P^{-1/2},$$

which gives

$$\begin{aligned}&\left(\left(1-t\right)P^{-1/2}AP^{-1/2}+tP^{-1/2}BP^{-1/2}+\lambda\right)^{-1}\\&=P^{1/2}\left(\left(1-t\right)A+tB+\lambda P\right)^{-1}P^{1/2}\end{aligned}$$

and by (2.9),

$$\begin{aligned}&f\left(P^{-1/2}BP^{-1/2}\right)-f\left(P^{-1/2}AP^{-1/2}\right)\\&=bP^{-1/2}\left(B-A\right)P^{-1/2}\\&+\int_0^\infty\lambda^2\left[\int_0^1P^{1/2}\left(\left(1-t\right)A+tB+\lambda P\right)^{-1}P^{1/2}P^{-1/2}\left(B-A\right)P^{-1/2}\right.\\&\left.\times P^{1/2}\left(\left(1-t\right)A+tB+\lambda P\right)^{-1}P^{1/2}dt\right]dw\left(\lambda\right)\\&=bP^{-1/2}\left(B-A\right)P^{-1/2}\\&+\int_0^\infty\lambda^2\left[\int_0^1P^{1/2}\left(\left(1-t\right)A+tB+\lambda P\right)^{-1}\left(B-A\right)\right.\\&\left.\times\left(\left(1-t\right)A+tB+\lambda P\right)^{-1}P^{1/2}dt\right]dw\left(\lambda\right).\end{aligned}\tag{2.10}$$

If we multiply both sides of (2.10) by $P^{1/2}$ we obtain the desired identity (2.8). □

Lemma 2.3. *Assume that the function $f:[0,\infty)\to\mathbb{R}$ is operator monotone in $[0,\infty)$ and has the representation (1.3). Then for all U, $V>0$ we have*

$$\begin{aligned}&\tilde{f}\left(V\right)-\tilde{f}\left(U\right)\\&=f\left(0\right)\left(V-U\right)+\int_0^\infty\lambda\left(\int_0^1\left(1+\lambda\left[\left(1-t\right)U+tV\right]\right)^{-1}\right.\\&\left.\times\left(V-U\right)\left(1+\lambda\left[\left(1-t\right)U+tV\right]\right)^{-1}dt\right)dw\left(\lambda\right).\end{aligned}\tag{2.11}$$

Proof. From (1.3) we have

$$f\left(t\right)=f\left(0\right)+bt+t\int_0^\infty\frac{\lambda}{t+\lambda}dw\left(\lambda\right),\ t>0.$$

If we put $\frac{1}{t}$ instead of t we get

$$\begin{aligned} f\left(\frac{1}{t}\right) &= f(0) + b\frac{1}{t} + \frac{1}{t}\int_0^\infty \frac{\lambda}{\frac{1}{t}+\lambda} dw(\lambda) \\ &= f(0) + b\frac{1}{t} + \frac{1}{t}\int_0^\infty \frac{t\lambda}{1+t\lambda} dw(\lambda) \end{aligned}$$

and by multiplication with $t > 0$, we get

$$\begin{aligned} \tilde{f}(t) &= b + tf(0) + \int_0^\infty \frac{t\lambda}{1+t\lambda} dw(\lambda) \\ &= b + tf(0) + \int_0^\infty \left(1 - \frac{1}{1+t\lambda}\right) dw(\lambda). \end{aligned}$$

Therefore

$$\tilde{f}(V) - \tilde{f}(U) = f(0)(V-U) + \int_0^\infty \left[(1+U\lambda)^{-1} - (1+V\lambda)^{-1}\right] dw(\lambda). \quad (2.12)$$

From (2.6) we get

$$\begin{aligned} &(1+U\lambda)^{-1} - (1+V\lambda)^{-1} \quad (2.13) \\ &= \int_0^1 ((1-t)(1+U\lambda) + t(1+V\lambda))^{-1} ((1+V\lambda) - (1+U\lambda)) \\ &\times ((1-t)(1+U\lambda) + t(1+V\lambda))^{-1} dt \\ &= \int_0^1 \lambda (1+\lambda[(1-t)U + tV])^{-1} (V-U)(1+\lambda[(1-t)U+tV])^{-1} dt. \end{aligned}$$

Therefore, by (2.12) we get

$$\begin{aligned} &\tilde{f}(V) - \tilde{f}(U) \quad (2.14) \\ &= f(0)(V-U) + \int_0^\infty \left[(1+U\lambda)^{-1} - (1+V\lambda)^{-1}\right] dw(\lambda) \\ &= f(0)(V-U) + \int_0^\infty \lambda \left(\int_0^1 (1+\lambda[(1-t)U+tV])^{-1} \right. \\ &\left. \times (V-U)(1+\lambda[(1-t)U+tV])^{-1} dt\right) dw(\lambda) \end{aligned}$$

and the identity (2.11) is proved. □

Theorem 2.4. *Assume that the function $f:(0,\infty)\to\mathbb{R}$ is operator monotone in $(0,\infty)$ and has the representation (1.3). Then for all C, D, $Q>0$ we have*

$$\begin{aligned}&\mathcal{P}_{\tilde{f}}(D,Q)-\mathcal{P}_{\tilde{f}}(C,Q) \\ &=a(D-C)+\int_0^\infty \lambda\left(\int_0^1 Q\left[\left(Q+\lambda\left[(1-t)C+tD\right]\right)\right]^{-1}(D-C)\right. \\ &\left.\times\left[\left(Q+\lambda\left[(1-t)C+tD\right]\right)\right]^{-1}Qdt\right)dw(\lambda).\end{aligned}\tag{2.15}$$

Proof. If we take $V=Q^{-1/2}DQ^{-1/2}$ and $U=Q^{-1/2}CQ^{-1/2}$ in (2.11), then we get

$$\begin{aligned}&\tilde{f}\left(Q^{-1/2}DQ^{-1/2}\right)-\tilde{f}\left(Q^{-1/2}CQ^{-1/2}\right) \\ &=f(0)\left(Q^{-1/2}DQ^{-1/2}-Q^{-1/2}CQ^{-1/2}\right) \\ &+\int_0^\infty \lambda\left(\int_0^1\left(1+\lambda\left[(1-t)Q^{-1/2}CQ^{-1/2}+tQ^{-1/2}DQ^{-1/2}\right]\right)^{-1}\right. \\ &\times\left(Q^{-1/2}DQ^{-1/2}-Q^{-1/2}CQ^{-1/2}\right) \\ &\left.\times\left(1+\lambda\left[(1-t)Q^{-1/2}CQ^{-1/2}+tQ^{-1/2}DQ^{-1/2}\right]\right)^{-1}dt\right)dw(\lambda)\end{aligned}\tag{2.16}$$

$$\begin{aligned}&=f(0)Q^{-1/2}(D-C)Q^{-1/2} \\ &+\int_0^\infty \lambda\left(\int_0^1\left[Q^{-1/2}\left(Q+\lambda\left[(1-t)C+tD\right]\right)\right]^{-1}\right. \\ &\left.\times Q^{-1/2}(D-C)Q^{-1/2}\left[Q^{-1/2}\left(Q+\lambda\left[(1-t)C+tD\right]\right)Q^{-1/2}\right]^{-1}dt\right)dw(\lambda) \\ &=f(0)Q^{-1/2}(D-C)Q^{-1/2} \\ &+\int_0^\infty \lambda\left(\int_0^1 Q^{1/2}\left[\left(Q+\lambda\left[(1-t)C+tD\right]\right)\right]^{-1}(D-C)\right. \\ &\left.\times\left[\left(Q+\lambda\left[(1-t)C+tD\right]\right)\right]^{-1}Q^{1/2}dt\right)dw(\lambda).\end{aligned}$$

If we multiply both sides by $Q^{1/2}$ we get the desired result (2.15). □

Corollary 2.5. *Assume that the function $f:[0,\infty)\to\mathbb{R}$ is operator monotone in*

$[0, \infty)$ *and has the representation (1.3). Then for all* C, D, $Q > 0$ *we have*

$$\begin{aligned} & \mathcal{P}_f(Q, D) - \mathcal{P}_f(Q, C) \qquad (2.17) \\ & = f(0)(D - C) + \int_0^\infty \lambda \left(\int_0^1 Q\left[(Q + \lambda [(1-t)C + tD]) \right]^{-1} (D - C) \right. \\ & \left. \times \left[(Q + \lambda [(1-t)C + tD]) \right]^{-1} Q dt \right) dw(\lambda). \end{aligned}$$

We also have:

Corollary 2.6. *Assume that the function* $f : (0, \infty) \to \mathbb{R}$ *is operator monotone in* $(0, \infty)$ *and has the representation (1.3). Then for all* A, B, C, $D > 0$ *we have*

$$\begin{aligned} & \mathcal{P}_f(A, B) - \mathcal{P}_f(C, D) \qquad (2.18) \\ & = b(A - C) + f(0)(B - D) \\ & + \int_0^\infty \lambda^2 \left[\int_0^1 B\left((1-t)C + tA + \lambda B \right)^{-1} (A - C) \right. \\ & \left. \times \left((1-t)C + tA + \lambda B \right)^{-1} B dt \right] dw(\lambda) \\ & + \int_0^\infty \lambda \left(\int_0^1 C\left[(C + \lambda [(1-t)D + tB]) \right]^{-1} (B - D) \right. \\ & \left. \times \left[(C + \lambda [(1-t)D + tB]) \right]^{-1} C dt \right) dw(\lambda). \end{aligned}$$

Proof. Observe that

$$\mathcal{P}_f(A, B) - \mathcal{P}_f(C, D) = \mathcal{P}_f(A, B) - \mathcal{P}_f(C, B) + \mathcal{P}_f(C, B) - \mathcal{P}_f(C, D). \qquad (2.19)$$

Since, by (2.8),

$$\begin{aligned} & \mathcal{P}_f(A, B) - \mathcal{P}_f(C, B) \qquad (2.20) \\ & = b(A - C) + \int_0^\infty \lambda^2 \left[\int_0^1 B\left((1-t)C + tA + \lambda B \right)^{-1} (A - C) \right. \\ & \left. \times \left((1-t)C + tA + \lambda B \right)^{-1} B dt \right] dw(\lambda) \end{aligned}$$

and by (2.17),

$$\begin{aligned} & \mathcal{P}_f(C, B) - \mathcal{P}_f(C, D) \qquad (2.21) \\ & = f(0)(B - D) + \int_0^\infty \lambda \left(\int_0^1 C\left[(C + \lambda [(1-t)D + tB]) \right]^{-1} (B - D) \right. \\ & \left. \times \left[(C + \lambda [(1-t)D + tB]) \right]^{-1} C dt \right) dw(\lambda), \end{aligned}$$

hence by (2.19)-(2.21) we obtain (2.18). □

As a natural consequence of the above representations, we derive the following inequalities:

Theorem 2.7. *Assume that the function* $f : [0, \infty) \to \mathbb{R}$ *is operator monotone in* $[0, \infty)$ *and has the representation (1.3). If* $B \geq A > 0$ *and* $P > 0$, *then*

$$\mathcal{P}_f(B, P) - \mathcal{P}_f(A, P) \geq b(B - A) \geq 0 \tag{2.22}$$

and

$$\mathcal{P}_f(P, B) - \mathcal{P}_f(P, A) \geq f(0)(B - A). \tag{2.23}$$

If $A \geq C > 0$ *and* $B \geq D > 0$, *then*

$$\mathcal{P}_f(A, B) - \mathcal{P}_f(C, D) \geq b(A - C) + f(0)(B - D). \tag{2.24}$$

Proof. If $B - A \geq 0$, then by multiplying both sides by $((1-t)A + tB + \lambda P)^{-1}$ for $t \in [0, 1]$ and $\lambda \geq 0$ we get

$$((1-t)A + tB + \lambda P)^{-1}(B - A)((1-t)A + tB + \lambda P)^{-1} \geq 0.$$

Also by multiplying both sides by $P > 0$, we get

$$P((1-t)A + tB + \lambda P)^{-1}(B - A)((1-t)A + tB + \lambda P)^{-1} P \geq 0,$$

for $t \in [0, 1]$ and $\lambda \geq 0$.

If we multiply this inequality by λ^2 integrate over $t \in [0, 1]$ and over the measure $w(\lambda)$ on $[0, \infty)$ we get

$$\int_0^\infty \lambda^2 \left[\int_0^1 P((1-t)A + tB + \lambda P)^{-1}(B - A) \times ((1-t)A + tB + \lambda P)^{-1} P dt\right] dw(\lambda) \geq 0$$

and by representation (2.8) we deduce (2.22).

The inequality (2.23) follows in a similar way by (2.17). The inequality (2.24) follows by the representation (3.2). □

Remark 2.8. If $f : [0, \infty) \to \mathbb{R}$ is operator monotone in $[0, \infty)$ and nonnegative, then

$$\mathcal{P}_f(P, B) - \mathcal{P}_f(P, A) \geq f(0)(B - A) \geq 0, \tag{2.25}$$

if $B \geq A > 0$ and $P > 0$.

If f is defined on $[0, \infty)$ and nonnegative, then the inequality (2.24) improves (1.5).

3 Some Examples of Interest

We also have identities for the *weighted operator geometric mean:*

Proposition 3.1. *For all A, B, $P > 0$ and $r \in (0,1]$ we have*

$$\begin{aligned} & P\sharp_r B - P\sharp_r A \\ &= \frac{\sin(r\pi)}{\pi} \int_0^\infty \lambda^{r+1} \left[\int_0^1 P\left((1-t)A + tB + \lambda P\right)^{-1} (B-A) \right. \\ & \left. \times \left((1-t)A + tB + \lambda P\right)^{-1} P dt \right] d\lambda. \end{aligned} \tag{3.1}$$

The proof follows by (2.8) and (1.1) for the measure $dw(\lambda) = \frac{\sin(r\pi)}{\pi} \lambda^{r-1} d\lambda$. The dual case follows by (2.17) and (1.1).

Proposition 3.2. *For all C, D, $Q > 0$ and $r \in (0,1]$ we have*

$$\begin{aligned} & D\sharp_r Q - C\sharp_r Q \\ &= \frac{\sin(r\pi)}{\pi} \int_0^\infty \lambda^r \left(\int_0^1 Q\left[\left(Q + \lambda\left[(1-t)C + tD\right]\right)\right]^{-1} (D-C) \right. \\ & \left. \times \left[\left(Q + \lambda\left[(1-t)C + tD\right]\right)\right]^{-1} Q dt \right) d\lambda. \end{aligned} \tag{3.2}$$

The following identity for the logarithmic function also holds:

Lemma 3.3. *For all U, $V > 0$ we have the identity:*

$$\begin{aligned} & \ln V - \ln U \\ &= \int_0^\infty \left(\int_0^1 \left(\lambda + (1-t)U + tV\right)^{-1} (V-U) \left(\lambda + (1-t)U + tV\right)^{-1} dt \right) d\lambda. \end{aligned} \tag{3.3}$$

Proof. We have from the representation of logarithm (1.2) that

$$\ln V - \ln U = \int_0^\infty \frac{1}{\lambda+1} \left[(V-1)(\lambda+V)^{-1} - (U-1)(\lambda+U)^{-1} \right] d\lambda \tag{3.4}$$

for U, $V > 0$.

Since

$$\begin{aligned} & (V-1)(\lambda+V)^{-1} - (U-1)(\lambda+U)^{-1} \\ &= V(\lambda+V)^{-1} - U(\lambda+U)^{-1} - \left((\lambda+V)^{-1} - (\lambda+U)^{-1} \right) \end{aligned}$$

and

$$\begin{aligned}
&V(\lambda+V)^{-1}-U(\lambda+U)^{-1}\\
&=(V+\lambda-\lambda)(\lambda+V)^{-1}-(U+\lambda-\lambda)(\lambda+U)^{-1}\\
&=1-\lambda(\lambda+V)^{-1}-1+\lambda(\lambda+U)^{-1}=\lambda(\lambda+U)^{-1}-\lambda(\lambda+V)^{-1},
\end{aligned}$$

hence

$$\begin{aligned}
&(V-1)(\lambda+V)^{-1}-(U-1)(\lambda+U)^{-1}\\
&=\lambda(\lambda+U)^{-1}-\lambda(\lambda+V)^{-1}-\left((\lambda+V)^{-1}-(\lambda+U)^{-1}\right)\\
&=(\lambda+1)\left[(\lambda+U)^{-1}-(\lambda+V)^{-1}\right]
\end{aligned}$$

and by (3.4) we get

$$\ln V-\ln U=\int_0^\infty\left[(\lambda+U)^{-1}-(\lambda+V)^{-1}\right]d\lambda. \tag{3.5}$$

Since, by (2.6) we have

$$\begin{aligned}
&(\lambda+U)^{-1}-(\lambda+V)^{-1}\\
&=\int_0^1(\lambda+(1-t)U+tV)^{-1}(V-U)(\lambda+(1-t)U+tV)^{-1}dt,
\end{aligned}\tag{3.6}$$

for all $\lambda\geq 0$, hence by (3.5) and (3.6) we get (3.3). □

Theorem 3.4. *For all A, B, $P>0$ we have*

$$\begin{aligned}
S(P|B)-S(P|A)=\int_0^\infty\Bigg[\int_0^1 & P((1-t)A+tB+\lambda P)^{-1}(B-A)\\
&\times((1-t)A+tB+\lambda P)^{-1}Pdt\Bigg]d\lambda.
\end{aligned}\tag{3.7}$$

Proof. Follows by Lemma 3.3 by taking $V=P^{-1/2}BP^{-1/2}$ and $U=P^{-1/2}AP^{-1/2}$ and multiplying both sides by $P^{1/2}$. □

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