Libertas Mathematica (new series)
Volume 37(2017), No. 2, 1-12

Selections of set-valued mappings via applications

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Abstract: Our aim is to study the problem of tightness of compact subsets of the space $M_r(X)$ of all Radon measures on the space X equipped by the topology of weak convergence. A kernel on a space Z into the space $M_r(S)$ is a continuous mapping $k : Z \longrightarrow M_r(X)$. A space X is called a uniformly Prohorov space if for each $\varepsilon > 0$, any paracompact space Z and any kernel $k : Z \longrightarrow M_r(X)$ there exists an upper semi-continuous compact-valued mapping $S_{(k,\varepsilon)} : Z \longrightarrow X$ such that $\mu_{(k,z)}(X \setminus S_{(k,\varepsilon)}(z)) \leq \varepsilon$ for each $z \in Z$. Any sieve-complete space is a uniformly Prohorov space (Corollary 3.4). Any uniformly Prohorov space is a Prohorov space. A space X is sieve-complete if and only if X is an open continuous image of a paracompact Čech-complete space. The idea of the concept of a uniformly Prohorov goes to A. Bouziad, V. Gutev and V. Valov.

Keywords: Set-valued mapping Selection, Radon measure, support of measure, Prohorov space

MSC2010: 28C15, 54C60, 54C35, 54C50

Dedicated to Academician Radu Miron on the occasion of his 90'th birthday

1 Introduction

By a space we understand a completely regular topological Hausdorff space. We use the terminology from [13]. Let βX denote the StoneÂ-Čech compactification of a space X, $cl_X A$ or clA denote the closure of a set A in a space X, $\mathbb{R}$ denote the space of reals, $\mathbb{N} = \{1, 2, ...\}$ denote the discrete space of natural numbers.

Let X be a space. By $Bo(X)$ we denote the σ-algebra of Borel subsets of X, by $C(X)$ we mean the ring of all real-valued continuous functions on X and by $C^0(X)$ we denote the Banach space of all continuous real-valued bounded functions on X

with the sup-norm $\|f\| = sup\{|f(x)| : x \in X\}$. Let $exp(X)$ be the family of all closed subsets of X and $exp_c(X)$ be the family of all compact subsets of X.

The space $exp(X)$ is endowed with the Vietoris topology. Recall that the Vietoris topology is generated by the open base of all collections of the form $e(\mathcal{V}) = \{F \in exp(X) : F \subset \cup\mathcal{V}, F \cap V \neq \emptyset \text{ } for \text{ } any \text{ } V \in \mathcal{V}\}$, where $\mathcal{V}$ runs over the finite families of open subsets of X. We consider $exp_c(X)$ as the subspace of the space $exp(X)$.

A *Borel measure* defined on the space X is a non-negative function $\mu : Bo(X) \longrightarrow \mathbb{R}$ with the following properties:

- $\mu(\emptyset) = 0$;

- If $A, B \in Bo(X)$ and $A \cap B = \emptyset$, then $\mu(A \cup B) = \mu(A) + \mu(B)$;

-$\mu(B) = sup\{\mu(F) : F \subset B, F \in exp(X)\}$ for every $B \in Bo(X)$.

Let μ be a Borel measure on X. Then for every function $f \in C^0(X)$ the integral $\int f d\mu$ is defined, and there exists a unique linear non-negative functional $u_\mu : C^0(X) \longrightarrow \mathbb{R}$ such that $u_\mu(f) = \int f d\mu$ for every $f \in C^0(X)$ (see [14, 21, 25, 18, 23, 24]). Equip the space $M(X)$ of all Borel measures with the topology of weak convergence. The weak topology on the space of all Borel measures $M(X)$ is the weakest topology for which $\mu_\alpha \to \mu$ if and only if $\int f d\mu_\alpha \to \int f d\mu$ for every $f \in C^0(X)$.

The measure μ is called:

(1) σ-additive if $\mu(\cup\{A_n : n \in \mathbb{N}\}) = \Sigma\{\mu(A_n) : n \in \mathbb{N}\}$ provided $\{A_n \in Ba(X) : n \in \mathbb{N}\}$ and $A_n \cap A_m = \emptyset$ for $n \neq m$;

(2) τ-additive or smooth if for any net of bounded continuous functions $\{f_\lambda \in C^0(X) : \lambda \in L\}$ which is monotone decreasing ($\lambda < \eta$ implies $f_\eta \leq f_\lambda$) and pointwise convergent to 0 we have $lim\{\int f_\lambda d\mu : \lambda \in L\}) = 0$;

(3) *tight* or *Radon measure* if $\mu(B) = sup\{\mu(F) : F \subset B, F \in exp_c(X)\}$ for every $B \in Bo(X)$.

If $M_\sigma(X)$ is the set of all σ-additive measures, $M_\tau(X)$ is the set of all τ-additive measures and $M_r X)$ is the set of all Radon measures on the space X, then $M_r(X) \subset M_\tau(X) \subset M_\sigma(X) \subset M(X)$. Denote by $P(X) = \{\mu \in M_r(X) : \mu(X) = 1\}$ the set of all *probabilist measures* on the space X.

For every Radon measure μ on X there exists a unique Radon measure $\beta\mu$ on βX such that $\beta\mu(H) = \mu(H \cap X)$ for every $H \in Bo(\beta X)$.

A subset Φ of the space $M(X)$ is called *tight* if for every $\varepsilon > 0$ there exists a compact subset $K_\varepsilon \subset X$ such that $\mu(X \setminus K_\varepsilon) < \varepsilon$ for each $\mu \in \Phi$. A space X is a *Prohorov space* if any compact subset of $M_r(X)$ is tight. Our aim is to study the problem of uniform tightness of compact subsets of the space $M_r(X)$.

The class of sieve-complete spaces was defined in [26, 4, 6, 10]. Let X be a space, $\gamma = \{\gamma_n = \{U_\alpha : \alpha \in A_n\} : n \in \mathbb{N}\}$ be a sequence of families of open non-empty subsets of the space X, and let $\pi = \{\pi_n : A_{n+1} \to A_n : n \in \omega\}$ be a sequence of single-valued mappings. A sequence $\alpha = \{\alpha_n : n \in \mathbb{N}\}$ is called *a spectral sequence* if $\alpha_n \in A_n$ and $\pi_n(\alpha_{n+1}) = \alpha_n$ for every $n \in \mathbb{N}$. Consider the following conditions:

(SC1) $\cup\{U_\beta : \beta \in A_n\} = X$ for each $n \in \mathbb{N}$.

(SC2) $\cup\{U_\beta : \beta \in \pi_n^{-1}(\alpha)\} = \cup\{cl_X U_\beta : \beta \in \pi_n^{-1}(\alpha)\} = U_\alpha$ for all $\alpha \in A_n$ and $n \in \mathbb{N}$.

(SC3) For each spectral sequence $\alpha = \{\alpha_n : n \in \mathbb{N}\}$ any set $cl_X\{x_n \in U_{\alpha_n} : n \in \mathbb{N}\}$ is a compact subset of the space X and the set $H(\gamma, \alpha) = \cap\{U_{\alpha_n} : n \in \mathbb{N}\}$ is a compact subset of the space X.

(SC4) For all $n \in \mathbb{N}$, $\alpha \in A_n$ and $\beta \in q^{-1}(\alpha)$ the sets $X \setminus U_\alpha$ and U_β form a pair of completely separated subsets of the space X.

The sequences γ and π are called *an A-sieve* if they are Properties (SC1) and (SC2). The sieve with property (SC4) is called a completely separated sieve. The sequences γ and π are called an *CA-sieve* if they are Properties (SC1), (SC2) and (SC3). If on the space X there exists a CA-sieve, then on X there exists a CA-sieve with Property (SC4) [6, 10].

A space X is called *sieve-complete* if on X there exists an CA-sieve (see [6, 10, 4]). Any Čech-complete space is sieve-complete.

We mention the following characteristic of sieve-complete spaces (see [6, 9]):

Corollary 1.1. *For a space X the following assertions are equivalent:*

1. *X is a sieve-complete space.*
2. *X is an open continuous image of a paracompact Čech-complete space.*
3. *$exp_c(X)$ is an open continuous image of a paracompact Čech-complete space.*
4. *$exp_c(X)$ is a sieve-complete space.*

Proof. The equivalences $1 \to 2 \to 1$ were proved in [6, 26]. Let $\varphi : Z \longrightarrow X$ be a continuous open mapping of a sieve-complete space Z onto X. Then for any open subset U of Z and any compact subset $F \subset \varphi(U)$ there exists a compact subset $\Phi \subset U$ such that $\varphi(\Phi) = F$. Consider the mapping $\varphi_c : exp_c(Z) \longrightarrow exp_c(X)$, where $\varphi_c(\Phi) = \varphi(\Phi)$ for each $\Phi \in exp_c(Z)$. The element $\Phi \in exp_c(Z)$ is considered as a point of the space $exp_c(Z)$ and as a subset of Z. By virtue of Theorem 2 from [9], φ_c is a continuous open mapping of $exp_c(Z)$ onto $exp_c(X)$. If Z is a paracompact Čech-complete space, then $exp_c(Z)$ is a paracompact Čech-complete space too ([7], Theorems 1 and 4). This proved the implications $2 \to 3 \to 4$. Since X is a closed subspace of the space $exp_c(X)$, the proof is complete. □

2 Support of measures

A correspondence $\theta : X \longrightarrow Y$ is called a set-valued mapping if $\theta(x) \in exp(Y)$ and $\theta(x) \neq \emptyset$ for each $x \in X$. We recall that a set-valued mapping $\theta : X \longrightarrow Y$

is lower (upper) semi-continuous, or l.s.c. (u.s.c), if the set $\theta^{-1}(H) = \{x \in X : \theta(x) \cap H \neq \emptyset\}$ is open (closed) in X for for every open (closed) subset H of Y. Let $\varphi, \psi : X \longrightarrow Y$ be set-valued mappings. If $\varphi(x) \subset \psi(x)$ for each $x \in X$, then φ is called a *selection* of the mapping ψ. If $\varphi(x) \cap \psi(x) \neq \emptyset$ for each $x \in X$, then φ is called a *generalized selection* of the mapping ψ. The articles [6, 10] contain some applications of generalized selections.

Let X be a space. For every Borel measure $\mu \in M(X)$ the support $supp_X(\mu)$ is the set of all points $x \in X$ such that $\mu(U) > 0$ for each neighbourhood U of the point x in X. Obviously, the set $supp_X(\mu)$ is a closed subset of X. Every point-finite family of open subsets of the subspace $supp_X(\mu)$ is countable (see [11, 7]). Moreover, the Souslin number (cellularity) $c(supp_X(\mu))$ is countable.

In ([15], Propositions 2.1 and 2.2) for the space $P(X)$ were proved the following important assertions:

Proposition 2.1. *Let X be a space, and let $\varepsilon \in (0,1)$. Define the set-valued mappings $\Psi_\varepsilon, \Phi_\varepsilon : M_r(X) \longrightarrow exp_c(X)$ by $\Psi_\varepsilon(\eta) = \{F \in \exp_c(X) : \eta(X \setminus F) < \varepsilon\}$ and $\Phi_\varepsilon(\eta) = cl_{exp_c(X)}\Psi_\varepsilon(\eta)$. Then:*

1. *Ψ_ε is a lower semi-continuous mapping.*
2. *Φ_ε is a lower semi-continuous mapping.*
3. *$\eta(X \setminus F) \leq \varepsilon$ for each $\eta \in M_r(X)$ and $F \in \Phi_\varepsilon(\eta)$.*

Proof. Let $\varepsilon \in (0,1)$, $\mathcal{V} = \{V_1, V_2, ..., V_n\}$ be a finite collection of non-empty subsets of X and $V = \cup\mathcal{V}$. Fix $\eta \in M_r(X)$. Assume that $K \in \Psi_\varepsilon(\eta) \cap e(\mathcal{V})$. Since $\eta(X) = p < +\infty$, $K \subset V$ and $\eta(X \setminus K) < \epsilon$, we have $\eta(X \setminus V) < \varepsilon$. Let $3\delta = \eta(K) + \varepsilon - p$. Consider the continuous functions $f, h : X \longrightarrow [0,1]$ with properties:

- $f(x) = 1$ for each $x \in K$ and $cl_X(X \setminus f^{-1}(0)) \subset V$;
- $h(x) = 1$ for each $x \in X$.

Then the set $U = \{\xi \in M_r(X) : p - \delta < \int hd\xi < p + \delta, \eta(K) - \delta < \int fd\xi < p + \delta\}$ is open in $M_r(X)$ and $\eta \in U$. We put $H = cl_X(X \setminus f^{-1}(0))$. Let $\xi \in U$. Since $p - \delta < \int hd\xi < p + \delta$, we have $p - \delta < \xi(X) < p + \delta$. Since $\eta(K) - \delta < \int fd\xi < p + \delta$, we have $\xi(U) > \eta(K) - \delta$. Hence $p - \varepsilon + \delta < \eta(K) - \delta < \xi(H) \leq \xi(X) < p + \delta$ and $\xi(H) > \xi(X) - \varepsilon$. There exists a compact subset F of X such that $F \subset H$ and $\xi(H) \geq \xi(F) > \xi(X) - \varepsilon$. For every $i \leq n$ fix a point $a_i \in V_i$. Put $\Phi = F \cup \{a_1, a_2, ..., a_n\}$. Then $\Phi \in \Psi_\varepsilon(\xi) \cap e(\mathcal{V})$. Hence $\eta \in U \subseteq \Psi_\varepsilon^{-1}(e(\mathcal{V}))$ and the set $\Psi_\varepsilon^{-1}(e(\mathcal{V}))$ is open. Assertion 1 is proved. Assertion 2 follows from Assertion 1.

Assume that $\eta \in M_r(X)$ and $K \in \Phi_\varepsilon(\eta)$. Suppose that $\eta(X \setminus K) > \varepsilon$. There exists a compact subset $F \subset X \setminus K$ such that $\eta(F) > \varepsilon$. Let $V = X \setminus F$. The set $e(V)$ is open in $exp_c(X)$ and $K \in e(V)$. Then there exists a compact set $\Phi \in e(V) \cap \Psi_\varepsilon(\eta)$. Since $\Phi \subset V$ and $\Phi \cap F = \emptyset$, we have $\eta(X \setminus \Phi) > \varepsilon$, a contradiction. Assertion 3 is proved. This complete the proof. $\square$

The assertions from the following proposition were proved in [25].

Proposition 2.2. *Let $\mu \in M_\tau(X)$. Then $\mu(supp_X(\mu)) = \mu(X)$ and for any family γ of open subsets of X there exists a countable subfamily ξ such that $\mu(\cup\xi) = \mu(\cup\gamma)$.*

If X is a metacompact space or a subparacompact space and $\mu \in M_\tau(X)$, then the subspace $supp_X(\mu)$ is Lindelöf ([25], Theorem 27 for a paracompact space X).

The following theorem for locally compact spaces and complete metrizable spaces was proved in [19].

Theorem 2.3. *Let X be a sieve-complete space. Then any Borel τ-additive measure on X is a Radon measure.*

Proof. Fix $\mu \in M(X)$. There exist a paracompact Čech-complete space Z and an open continuous mapping $g : Z \longrightarrow X$. Fix a sequence $\{U_n : n \in \mathbb{N}\}$ of open subsets of βZ such that $Z = \cap\{U_n : n \in \mathbb{N}\}$ and $\epsilon > 0$. Let $n \in \mathbb{N}$, $\delta > 0$ and U is an open subset of Z. By virtue of Proposition 2.2, there exists an open subset $V(n, U, \delta)$ of Z such that $V(n, U, \delta) \subset U$, $cl_{\beta Z}V(n, U, \delta) \subset U_n$ and $\mu(g(U)) \setminus \mu(V(n, U, \delta)) < \delta$. Therefore, there exists a sequence $\{V_n : n \in \mathbb{N}\}$ of open subsets of βZ such that $V_{n+1} \subset cl_{\beta Z}V_{n+1} \subset V_n \subset cl_{\beta Z}V_n \subset U_n$ and $\mu(g(Z \cap V_n)) > \mu(X) - \epsilon$ for each $n \in \mathbb{N}$. Then $F = \cap\{V_n : n \in \mathbb{N}\}$ is a compact subset of Z and $g(F) = \cap\{g(Z \cap V_n) : n \in \mathbb{N}\}$. Hence $\mu(X \setminus g(F)) \leq \epsilon$. □

3 Uniformly Prohorov spaces

Let X and Z be topological spaces. A *measurable kernel* with source Z and target X is a mapping $\lambda : Z \times Bo(X) \longrightarrow \mathbb{R}$ with the following properties:

(1) $\mu_{(\lambda,z)} : Bo(X) \longrightarrow \mathbb{R}$, where $\mu_{(\lambda,z)}(A) = \lambda(z, A)$ for each $A \in Bo(X)$, is a Radon measure on X for each $z \in Z$;

(2) $\chi_{(\lambda,z)} : Z \longrightarrow \mathbb{R}$, where $\chi_{(\lambda,z)}(A) = \lambda(z, A)$ for each $z \in Z$, is a Borel measurable function on Z for each $A \in Bo(X)$;

(3) The mapping $k_\lambda : Z \longrightarrow M_r(X)$, where $\kappa_\lambda(z) = \mu_{(\lambda,z)}$ for each $z \in Z$, is continuous.

We say that a *kernel* on Z into the space $M_r(X)$ of Radon measures on X is a continuous mapping $k : Z \longrightarrow M_r(X)$. If $k : Z \longrightarrow M_r(X)$ is a kernel, then $\mu_(k, z) = k(z) \in M_r(X)$ [22].

A space X is called a *uniformly Prohorov space* if for each $\varepsilon > 0$ any paracompact space Z and any kernel $k : Z \longrightarrow M_r(X)$ there exists an upper semi-continuous compact-valued mapping $S_{(k,\varepsilon)} : Z \longrightarrow X$ such that $\mu_{(k,z)}(X \setminus S_{(k,\varepsilon)}(z)) \leq \varepsilon$ for each $z \in Z$.

The idea of the concept of a uniformly Prohorov goes to the articles of A. Bouziad [3] V. Gutev and V. Valov [15]. That is a "continuous version" of the notion of Prohorov space [21, 15].

Proposition 3.1. *Let X be a uniformly Prohorov space. Then X is a Prohorov space.*

Proof. Let Φ be a compact subset of the space $M_r(X)$ and $\varepsilon > 0$. Consider the identical mapping $k : \Phi \longrightarrow M_r(X)$, where $k(\mu) = \mu$ for each $\mu \in \Phi$. The space Φ is compact and there exists an upper semi-continuous compact-valued mapping $S_\varepsilon : \Phi \longrightarrow X$ such that $\mu(X \setminus S_\varepsilon(\mu)) \leq \varepsilon$ for each $\mu \in \Phi$. Then $F = S_\epsilon(\Phi)$ is a compact subset of X and $\mu(X \setminus F) \leq \mu(X \setminus S_\varepsilon(\mu)) \leq \varepsilon$ for each $\mu \in \Phi$. The proof is complete. $\square$

For the mapping $S_\varepsilon : P(X) \longrightarrow X$ the following theorem was proved in ([15].

Theorem 3.2. *Let X be a sieve-complete space. Then for any paracompact space Z, any kernel $k : Z \longrightarrow M_r(X)$ and any $\varepsilon > 0$ there exists an upper semi-continuous compact-valued mapping $S_{(k,\varepsilon)} : Z \longrightarrow X$ such that $\mu_{(k,z)}(X \setminus S_{(k,\varepsilon)}(z) \leq \varepsilon$ and $S_{(k,\varepsilon)}(F)$ is a compact of countable character in X for each compact subset F of Z.*

Proof. Consider the lower semi-continuous set-valued mappings $\Phi_\varepsilon : M_r(X) \longrightarrow exp_c(X)$ defined in Proposition 2.1. We have $\eta(X \setminus F) \leq \varepsilon$ for each $\eta \in M_r(X)$ and $F \in \Phi_\varepsilon(\eta)$. By virtue of Corollary 1.1, the space $exp_c(X)$ is sieve-complete. Consider now the lower semi-continuous set-valued mapping $\theta : Z \longrightarrow exp_c(X)$, where $\theta(z) = \Phi_\varepsilon(k(z)$ for each $z \in Z$.

As was proved in [6], Theorem 8.4, there exists an upper semi-continuous compact-valued mapping $G_\varepsilon : Z \longrightarrow exp_c X$ such that $G_\varepsilon(z) \cap \theta(z) \neq \emptyset$ for each $z \in Z$. From the construction of the mapping S_ε, it follows that the set and $S_\varepsilon(\eta)$ is a compact of countable character in X for each $\eta \in M_r(X)$. We put $S_{(k,\varepsilon)}(z) = \cup\{F \subset X : F \in G_\varepsilon(z)\}$ for each $z \in Z$. The mapping $S_{(k,\varepsilon)}$ is compact-valued (see [8], Lemma 4). Let U be an open subset of X. Then the set $V = \{F \in exp_c(X) : F \subset U\}$ is open in $exp_c(X)$. Hence the mapping $S_{(k,\varepsilon)}$ is upper semi-continuous. If $z \in Z$ and $F \in G_\varepsilon(z) \cap \Phi_\varepsilon(z)$, then $F \subset S_\varepsilon(\mu_{(k,z)})$ and $\mu_{(k,z)}(X \setminus S_{(k,\varepsilon)}(z)) \leq \mu_{(k,z)}(X \setminus F) \leq \varepsilon$. The proof is complete. $\square$

The space X is a paracompact Čech-complete space if and only if $M_r(X)$ is a paracompact Čech-complete space (see [2, 23, 7]). Hence, from Theorem 3.2 it follows:

Corollary 3.3. *Let X be a paracompact Čech-complete space. Then for any $\epsilon > 0$ there exists a upper semi-continuous compact-valued mapping $S_\varepsilon : M_r(X) \longrightarrow X$*

such that $\mu(X \setminus S_\varepsilon(\mu) \leq \varepsilon$ *and* $S_\varepsilon(F)$ *is a compact of countable character in* X *for each compact subset* F *of* $M_r(X)$*.*

Corollary 3.4. *Any sieve-complete space is a uniformly Prohorov space.*

Corollary 3.5. *[7] Any sieve-complete space is a Prohorov space.*

Example 3.6. Let X be a space. A typical distance between measures is of the form

$$\rho_{\mathcal{D}}(\mu,\nu) = sup\{|\int f d\mu - \int f d\nu| : f \in \mathcal{D}\},$$

where $\mathcal{D}$ is some class of Borel measurable functions on the space X.

The total variation distance between two measures μ and ν on a sigma-algebra $Bo(X)$ of subsets of the space X is defined via

$$\delta(\mu,\nu) = sup\{|\mu(A)–\nu(A)| : A \in Bo(X)\}.$$

If $\mathcal{D}_1 = \{1_A : A \in Bo(X)\}$, then $\delta(\mu,\nu) = sup\{|\int f d\mu - \int f d\nu| : f \in \mathcal{D}_1\}$ and $\delta = \rho_{\mathcal{D}_1}$.

In probability theory, the total variation distance is a distance measure for probability distributions. It is an example of a statistical distance metric, and is sometimes just called "the" statistical distance. If μ and ν are both probability measures, then the total variation distance is also given by

$$\| \mu - \nu \|_T V = 2 \cdot sup\{|\mu(A)–\nu(A)| : A \in Bo(X)\}.$$

Let $\mathcal{D}_2$ be the family of all Borel measurable functions $f : X \to [-1,1]$. Then the distance $\rho_{\mathcal{D}_2}(\mu,\nu)$ also is called the total variation distance between two measures μ and ν.

Radon distance between two measures μ and ν to be

$$d_r(\mu,\nu) = sup\{|\int f d\mu - \int f d\nu| : f \in C(X,[-1,1])\}.$$

For Radon measures μ and ν we have

$$\delta(\mu,\nu) = d_r(\mu,\nu) = \rho_{\mathcal{D}_2}(\mu,\nu).$$

If μ and $\{\mu_n : n \in \omega\}$ are Radon measures and $lim_{n\to\infty} d_r(\mu,\mu_n) = 0$, then $\mu_n \longrightarrow \mu$ in the weak topology too. Hence the space $(M_r(X), d_r)$ with the identical mapping k of $(M_r(X), d_r)$ onto $M_r(X)$ is a kernel on $(M_r(X), d_r)$ and the space $(M_r(X), d_r)$ is paracompact as a metric space.

4 On scattered spaces

A point $x \in X$ is called *a point of countable type* if there exists a compact subset F with a countable base of open neighborhoods $\{U_n : n \in \mathbb{N}\}$ in X such that $x \in F$. A space X is called a space of *pointwise countable type* if each $x \in X$ is a point of countable type [1, 13].

A space X is called a space of *countable type* if for each compact subset F of X there exists a compact subset H with a countable base of open neighborhoods $\{U_n : n \in \mathbb{N}\}$ in X such that $F \subseteq H$ [1, 13, 16]. Any sieve-complete space is of countable type [6].

A space X is called *a scattered* space if any non-empty subspace Y of X contains some isolated point in Y.

Proposition 4.1. *Let X be a space, $X \subset Y \subset \mu X$, Y be a space of pointwise countable type and a paracompact space Z is scattered or a union of countable family of closed discrete subspaces. Then for each lower semi-continuous mapping $\theta : Z \longrightarrow X$ there exists an upper semi-continuous compact-valued mapping $\varphi : Z \longrightarrow Y$ with the following properties:*

1. $\psi(z) = \varphi(z) \cap \theta(z) \neq \emptyset$ for any $z \in Z$.

2. If F is a compact subset of Z, then $\Phi = \varphi(F)$ is a compact of countable character in Y, $\Phi \cap X$ is a bounded subset of X and $\Phi = cl_Y(\Phi \cap X)$.

3. If U is a functionally open subset of Y and $V = U \cap X$, then $\{z \in Z : \varphi(z) \subset U\} = \{z \in Z : \psi(z) \subset V\} = \{z \in Z : \psi(z) \subset U\}$.

Proof. The case when Z is a union of countable family of closed discrete subspaces follows from results in [5, 6].

Assume that Z is a paracompact scattered space. Then $dim Z = 0$. It is sufficient to prove that for any point $b \in Z$ and any open subset H of Y for which $\theta(z) \cap H \neq \emptyset$, there exist an open-and-closed subspace $Z(b, H)$ and such an upper semi-continuous compact-valued mapping $\varphi_{(b,H)} : Z(b, H) \longrightarrow Y$ with the following properties:

1. $\varphi_{(b,H)}(z) \cap \theta(z) \neq \emptyset$ and $\varphi_{(b,H)}(z) \subset H$ for any $z \in Z$.

2. If F is a compact subset of $Z(b, H)$, then $\Phi = \varphi_{(b,H)}(F)$ is a compact of countable character in Y, $\Phi \cap X$ is a bounded subset of X and $\Phi = cl_Y(\Phi \cap X)$.

Fix a point $b \in Z$ and an open subset H of Y for which $\theta(z) \cap H \neq \emptyset$. If $b \in Z_(1)$, then we put $Z(b, H) = \{b\}$. Then we fix a point $x(b) \in X \cap \theta(z) \cap H$ and a compact Φ of countable character in Y such that $x(b) \in \Phi$, and put $\varphi_{(b,H)}(b) = \Phi$.

Assume that $\alpha \geq 2$ and for any point $b \in \cup\{Z_{(\beta)} : \beta < \alpha\}$ the objects $Z(b, H)$ and $\varphi_{(b,H)}$ there exist provided H is open in Y and $\theta(z) \cap H \neq \emptyset$.

Fix $b \in Z_{(\alpha)}$ and an open in Y subset for which $\theta(z) \cap H \neq \emptyset$. Then we fix a point $x(b) \in X \cap \theta(z) \cap H$ and a compact Φ of countable character in Y such that

$x(b) \in \Phi$. Let $\{U_n : n \in \mathbb{N}\}$ be a base of Φ in Y and $\{V_n : n \in \mathbb{N}\}$ be a sequence of open-and-closed subsets of Z such that:

- $b \in Z(b, H) = V_1 \subset \theta^{-1}(U_1)$;

- $U_{n+1} \subset U_n \subset H$ and $b \in V_{n+1} \subset V_n \subset \theta^{-1}(U_n$ for each $n \in \mathbb{N}$.

We put $B = \cap\{V_n : n \in \mathbb{N}\}$. Obviously that $\theta(z) \cap \Phi \neq \emptyset$ for any $z \in B$. Denote $B_n = V_n \setminus V_{n+1}$. Then $Z(b, H = V_1 = B \cup (\cup\{B_n : n \in \mathbb{N}\}$. Assume that $z \in Z(b, H)$. If $z \in B$, then $\varphi_{(b,H)}(z) = \Phi$. Suppose that $z \in B_n$ and $n \in \mathbb{N}$. Since B_n is a paracompact space and $dimZ_n = 0$, there exist a subset $A_n \subset B_n$ and a discrete cover $\{Wx : x \in A_n\}$ of the subspace B_n such that $Wx \subset Z(x, U_n)$ for each $x \in A_n$. We put $\varphi_{(b,H)}(z) = \varphi_{(x,U_n)}(z)$ for all $z \in Wx$ and $x \in A_n$. The objects $Z(b, H)$ and $\varphi_{(b,H)}$ are the desired set and mapping. The proof is complete. □

Corollary 4.2. *Let X be a space of countable type. Then:*

1. Any scattered compact subset of the space $M_r(X)$ is uniformly tight.

2. If Z is a paracompact scattered or a paracompact F_σ-discrete subspace of $M_r(X)$, then for any $\varepsilon > 0$ there exists an upper semi-continuous compact-valued mapping $S_\varepsilon : Z \longrightarrow X$ such that:

- $\eta(X \setminus S_\varepsilon(\eta) \leq \varepsilon$ and $S_\varepsilon(\eta)$ is a compact of countable character in X for each $\eta \in M_r(X)$;

- the set $S_\varepsilon(\Phi)$ is compact for any compact subset Φ of Y.

The assertion 1 of Corollary 4.2 was proved in [17, 23] and for metric spaces - in [12].

In [20] was proved that the space of rational numbers $\mathbb{Q}$ is not a Prohorov space.

Let $S = \{0\} \cup \{n{-}1 : n \in \mathbb{N}\}$ be the convergent sequence as a subspace of the closed interval $[0, 1]$. In [11, 7] was proved that there exist a non-Prohorov space Y, a Prohorov space X, an open continuous compact mapping $\varphi : Y \longrightarrow S$ of Y onto S and an open continuous compact mapping $\psi : X \longrightarrow Y$ of X onto Y such that:

(1) There exist $a \in X$ and $b \in Y$ such that $\varphi^{-1}(0) = \{b\}$ and $\psi^{-1}(b) = \{a\}$;

(2) Every compact subset of X is finite;

(3) Every compact subset of Y is finite;

(4) The subspaces $X \setminus \{a\}$ and $Y \setminus \{b\}$ are discrete and countable;

(5) The spaces S, X, Y are scattered.

Similar example was constructed in [3].

In [23] F. Topsoe raised the problem: Is a continuous open image of a Prohorov space a Prohorov space? Hence, in general, the answer is not "yes". Nevertheless the next problems which were formulated in [11, 7] remain open.

Question 1. Is a continuous open image of a metric Prohorov space a Prohorov space?

Question 2. Is a continuous open compact image of a first countable Prohorov space a Prohorov space?

Question 3. Is a continuous closed image of a metric Prohorov space a Prohorov space?

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