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Golden warped product Riemannian manifolds

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Dedicated to Academician Radu Miron on the occasion of his 90'th birthday

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1 Introduction

The notion of *Golden number* is provided by the positive solution of the equation $x^2 - x - 1 = 0$ and it has the form:

$$\sigma = \frac{1+\sqrt{5}}{2}. \tag{1.1}$$

Starting from a polynomial structure, which was generally defined by S. I. Goldberg, K. Yano and N. C. Petridis in ([7] and [8]), we consider a polynomial structure on an m-dimensional Riemannian manifold (M, g), called by us a *Golden structure* ([5], [10], [6] and [11]), determined by a $(1,1)$-tensor field J which satisfies the equation:

$$J^2 = J + I, \tag{1.2}$$

where I is the identity operator on the Lie algebra of vector fields on M identified with the set of smooth sections $\Gamma(T(M))$ (and we'll simply denote $X \in T(M)$).

Remark that a Golden structure J verifies the recurrence relation:

$$J^{n+1} = f_{n+1} \cdot J + f_n \cdot I, \tag{1.3}$$

where $(f_n)_{n\in\mathbb{N}}$ is the Fibonacci sequence defined by $f_{n+2} = f_{n+1} + f_n$, $f_1 = f_2 = 1$.

We say that the metric g is J-compatible if the following equality holds:

$$g(JX, Y) = g(X, JY). \tag{1.4}$$

If (M, g) is a Riemannian manifold endowed with a Golden structure J such that the Riemannian metric g is J-compatible, then (M, g, J) is called a *Golden Riemannian manifold.*

Also, we can remark that:

$$g(JX, JY) = g(X, JY) + g(X, Y), \tag{1.5}$$

for any $X, Y \in T(M)$.

We proved that an almost product structure F on M induces two Golden structures on M ([12]):

$$J_1 = \frac{1}{2} \cdot I + \frac{2\sigma - 1}{2} \cdot F, \quad J_2 = \frac{1}{2} \cdot I - \frac{2\sigma - 1}{2} \cdot F. \tag{1.6}$$

Conversely, every Golden structure J on M induces two almost product structures on M:

$$F_{\pm} = \pm \left(\frac{2}{2\sigma - 1} \cdot J - \frac{1}{2\sigma - 1} \cdot I \right). \tag{1.7}$$

In particular, if the almost product structure F is compatible with the Riemannian metric, then J_1 and J_2 are Golden Riemannian structures.

On a Golden manifold (M, J) there are two complementary distributions $\mathcal{D}_l$ and $\mathcal{D}_m$ corresponding to the projection operators l and m ([12]) given by:

$$l = \frac{\sigma}{2\sigma - 1} \cdot I - \frac{1}{2\sigma - 1} \cdot J, \quad m = \frac{\sigma - 1}{2\sigma - 1} \cdot I + \frac{1}{2\sigma - 1} \cdot J. \tag{1.8}$$

Moreover, the operators l and m verify the following equalities:

$$l + m = I, \quad l^2 = l, \quad m^2 = m, \quad lm = ml = 0, \tag{1.9}$$

$$Jl = lJ = (1 - \sigma)l, \quad Jm = mJ = \sigma m. \tag{1.10}$$

The analogue concept of locally product manifold is considered in the context of Golden geometry. We say that the Golden Riemannian manifold (M, g, J) is *locally Golden* if J is parallel with respect to the Levi-Civita connection associated to g.

2 Golden warped product Riemannian manifolds

2.1 Warped product manifolds

Consider (M_1, g_1) and (M_2, g_2) two Riemannian manifolds of dimensions n and m, respectively. Denote by p_1 and p_2 the projection maps from the product manifold $M_1 \times M_2$ to M_1 and M_2 and by $\widetilde{\varphi} := \varphi \circ p_1$ the lift to $M_1 \times M_2$ of a smooth function φ on M_1.

In this context, we shall call M_1 *the base* and M_2 *the fiber* of $M_1 \times M_2$, the unique element $\widetilde{X}$ of $T(M_1 \times M_2)$ that is p_1-related to $X \in T(M_1)$ and to the zero vector field on M_2, the *horizontal lift of* X and the unique element $\widetilde{V}$ of $T(M_1 \times M_2)$ that is p_2-related to $V \in T(M_2)$ and to the zero vector field on M_1, the *vertical lift of* V. Also denote by $\mathcal{L}(M_1)$ the set of all horizontal lifts of vector fields on M_1 and by $\mathcal{L}(M_2)$ the set of all vertical lifts of vector fields on M_2.

Let $f > 0$ be a smooth function on M_1 and

$$\widetilde{g} := p_1^* g_1 + (f \circ p_1)^2 p_2^* g_2 \tag{2.1}$$

be a Riemannian metric on $M_1 \times M_2$.

Definition 2.1. ([4]) The product manifold of M_1 and M_2 together with the Riemannian metric $\widetilde{g}$ defined by (2.1) is called *the warped product* of M_1 and M_2 by the warping function f [and it is denoted by $(\widetilde{M} := M_1 \times_f M_2, \widetilde{g})$].

Remark that if f is constant (equal to 1), the warped product becomes the usual product of the Riemannian manifolds.

For $(x, y) \in \widetilde{M}$, we shall identify $X \in T(M_1)$ with $(X_x, 0_y) \in T_{(x,y)}(\widetilde{M})$ and $Y \in T(M_2)$ with $(0_x, Y_y) \in T_{(x,y)}(\widetilde{M})$ ([3]).

Let $\pi_1 =: Tp_1$ and $\pi_2 =: Tp_2$ be the projection mappings of $T(M_1 \times M_2)$ into $T(M_1)$ and $T(M_2)$, respectively. They verify:

$$\pi_1 + \pi_2 = I, \quad \pi_1^2 = \pi_1, \quad \pi_2^2 = \pi_2, \quad \pi_1 \circ \pi_2 = \pi_2 \circ \pi_1 = 0. \tag{2.2}$$

The Riemannian metric of the warped product manifold $\widetilde{M} = M_1 \times_f M_2$ equals to $\widetilde{g}(\widetilde{X}, \widetilde{Y}) = g_1(\pi_1 \widetilde{X}, \pi_1 \widetilde{Y}) + (f \circ p_1)^2 g_2(\pi_2 \widetilde{X}, \pi_2 \widetilde{Y})$, thus:

$$\widetilde{g}(\widetilde{X}, \widetilde{Y}) = g_1(X_1, Y_1) + (f \circ p_1)^2 g_2(X_2, Y_2), \tag{2.3}$$

for any $\widetilde{X} = (X_1, X_2), \widetilde{Y} = (Y_1, Y_2) \in T(\widetilde{M}) = T(M_1 \times_f M_2)$. From the definition of $\widetilde{g}$, one can verify that the leaves $M_1 \times \{y\}$, for $y \in M_2$, are totally geodesic submanifolds of $(\widetilde{M} = M_1 \times_f M_2, \widetilde{g})$.

If we denote by $\widetilde{\nabla}$, ${}^{M_1}\nabla$, ${}^{M_2}\nabla$ the Levi-Civita connections on $\widetilde{M}$, M_1 and M_2, we know that for any $X_1, Y_1 \in T(M_1)$ and $X_2, Y_2 \in T(M_2)$ ([13]):

$$\widetilde{\nabla}_{(X_1,X_2)}(Y_1,Y_2) = ({}^{M_1}\nabla_{X_1}Y_1 - \frac{1}{2}g_2(X_2,Y_2)\cdot grad(f^2), \tag{2.4}$$

$${}^{M_2}\nabla_{X_2}Y_2 + \frac{1}{2f^2}X_1(f^2)Y_2 + \frac{1}{2f^2}Y_1(f^2)X_2).$$

Remark 2.2. For the case of product Riemannian manifolds:

i) the Riemannian curvature tensors verify ([2]):

$$R(\widetilde{X},\widetilde{Y})\widetilde{Z} = (R_1(X_1,Y_1)Z_1, R_2(X_2,Y_2)Z_2), \tag{2.5}$$

for any $\widetilde{X} = (X_1,X_2), \widetilde{Y} = (Y_1,Y_2), \widetilde{Z} = (Z_1,Z_2) \in T(M_1 \times M_2)$, where R, R_1 and R_2 are respectively the Riemannian curvature tensors of the Riemannian manifolds $(M_1 \times M_2, \widetilde{g})$, (M_1, g_1) and (M_2, g_2);

ii) the Ricci curvature tensors verify ([2]):

$$S(\widetilde{X},\widetilde{Y}) = S_1(X_1,Y_1) + S_2(X_2,Y_2), \tag{2.6}$$

for any $\widetilde{X} = (X_1,X_2), \widetilde{Y} = (Y_1,Y_2) \in T(M_1 \times M_2)$, where S, S_1 and S_2 are respectively the Ricci curvature tensors of the Riemannian manifolds $(M_1 \times M_2, \widetilde{g})$, (M_1, g_1) and (M_2, g_2).

Remark that the Riemannian curvature tensor of a locally Golden Riemannian manifold has the following properties:

Proposition 2.3. *If (M, g, J) is a locally Golden Riemannian manifold, then for any $X, Y, Z \in T(M)$:*

$$R(X,Y)JZ = J(R(X,Y)Z), \tag{2.7}$$

$$R(JX,Y) = R(X,JY), \tag{2.8}$$

$$R(JX,JY) = R(JX,Y) + R(X,Y), \tag{2.9}$$

$$R(J^{n+1}X,Y) = f_{n+1}\cdot R(JX,Y) + f_n \cdot R(X,Y), \tag{2.10}$$

where $(f_n)_{n\in\mathbb{N}}$ is the Fibonacci sequence defined by $f_{n+2} = f_{n+1} + f_n$, $f_1 = f_2 = 1$.

Proof. The locally Golden condition $\nabla J = 0$ is equivalent to $\nabla_X JY = J(\nabla_X Y)$, for any $X, Y \in T(M)$ and (2.7) follows from the definition of R. The relations (2.8), (2.9) and (2.10) follows from the symmetries of R and from the recurrence relation $J^{n+1} = f_{n+1}\cdot J + f_n \cdot I$. □

Let S, S_{M_1}, S_{M_2} be the Ricci curvature tensors on $\widetilde{M}$, M_1 and M_2 and $\widetilde{S_{M_1}}$, $\widetilde{S_{M_2}}$ the lift on $\widetilde{M}$ of S_{M_1} and S_{M_2}. Then:

Lemma 2.4. *([4]) If $(\widetilde{M} := M_1 \times_f M_2, \widetilde{g})$ is the warped product of M_1 and M_2 by the warping function f and $m > 1$, then for any X, $Y \in \mathcal{L}(M_1)$ and any V, $W \in \mathcal{L}(M_2)$, we have:*

1. $S(X,Y) = \widetilde{S_{M_1}}(X,Y) - \frac{m}{f}H^f(X,Y)$, *where H^f is the lift on $\widetilde{M}$ of $Hess(f)$;*
2. $S(X,V) = 0$;
3. $S(V,W) = \widetilde{S_{M_2}}(V,W) - \left[\frac{\Delta(f)}{f} + (m-1)\frac{|grad(f)|^2}{f^2}\right] g(V,W)$.

2.2 Golden warped product Riemannian manifolds

i) Golden Riemannian structure on $(\widetilde{M}, \widetilde{g})$ induced by the projection operators

The endomorphism

$$F = \pi_1 - \pi_2 \tag{2.11}$$

verifies $F^2 = I$ and $\widetilde{g}(F\widetilde{X}, \widetilde{Y}) = \widetilde{g}(\widetilde{X}, F\widetilde{Y})$, thus F is an almost product structure on $M_1 \times M_2$.

By using relations (1.6) we can construct on $M_1 \times M_2$ two Golden structures, given by:

$$\widetilde{J}_\pm = \frac{I \pm \sqrt{5}F}{2}. \tag{2.12}$$

Also from $\widetilde{g}(F\widetilde{X}, \widetilde{Y}) = \widetilde{g}(\widetilde{X}, F\widetilde{Y})$ follows $\widetilde{g}(\widetilde{J}_\pm\widetilde{X}, \widetilde{Y}) = \widetilde{g}(\widetilde{X}, \widetilde{J}_\pm\widetilde{Y})$. Therefore, we can state the following result:

Theorem 2.5. *There exists two Golden Riemannian structure $\widetilde{J}_\pm$ on $(\widetilde{M}, \widetilde{g})$ given by:*

$$\widetilde{J}_\pm = \frac{I \pm \sqrt{5}F}{2}, \tag{2.13}$$

where $\widetilde{M} = M_1 \times_f M_2$ and $\widetilde{g}(\widetilde{X}, \widetilde{Y}) = g_1(X_1, Y_1) + (f \circ p_1)^2 g_2(X_2, Y_2)$, for any $\widetilde{X} = (X_1, X_2), \widetilde{Y} = (Y_1, Y_2) \in T(\widetilde{M}) = T(M_1 \times_f M_2)$.

We remark that for $\widetilde{J}_+ = \frac{I+\sqrt{5}F}{2}$, the projection operators are $\pi_1 = m$, $\pi_2 = l$ and for $\widetilde{J}_- = \frac{I-\sqrt{5}F}{2}$ we have $\pi_1 = l$, $\pi_2 = m$, where m and l were given in relations (1.8).

Remark 2.6. If we denote by $\widetilde{\nabla}$ the Levi-Civita connection on $\widetilde{M}$ with respect to $\widetilde{g}$, we can check that $\widetilde{\nabla}F = 0$ [hence $\widetilde{\nabla}\widetilde{J}_{\pm} = 0$ and so $(\widetilde{M} = M_1 \times_f M_2, \widetilde{g}, \widetilde{J}_{\pm})$ is a locally Golden Riemannian manifold].

For the case of product Riemannian manifolds, from (2.5) and Proposition 2.3 we have:

Proposition 2.7. *Let* $(\widetilde{M} = M_1 \times M_2, \widetilde{g}, \widetilde{J}_{\pm})$ *(with* $\widetilde{g}$ *given by* (2.1) *for* $f = 1$ *and* $\widetilde{J}_{\pm}$ *given by* (2.12)*) be the product of the Riemannian manifolds* (M_1, g_1) *and* (M_2, g_2) *with the Golden structure* $\widetilde{J}_{\pm}$*. Then, for any* $\widetilde{X}, \widetilde{Y}, \widetilde{Z} \in T(\widetilde{M}) = T(M_1 \times M_2)$*, the Riemannian curvature tensor verifies the relations:*

$$R(\widetilde{X}, \widetilde{Y})\widetilde{J}_{\pm}\widetilde{Z} = \widetilde{J}_{\pm}(R(\widetilde{X}, \widetilde{Y})\widetilde{Z}), \tag{2.14}$$

$$R(\widetilde{J}_{\pm}\widetilde{X}, \widetilde{Y}) = R(\widetilde{X}, \widetilde{J}_{\pm}\widetilde{Y}), \tag{2.15}$$

$$R(\widetilde{J}_{\pm}\widetilde{X}, \widetilde{J}_{\pm}\widetilde{Y}) = R(\widetilde{J}_{\pm}\widetilde{X}, \widetilde{Y}) + R(\widetilde{X}, \widetilde{Y}), \tag{2.16}$$

$$R(\widetilde{J}_{\pm}^{n+1}\widetilde{X}, \widetilde{Y}) = f_{n+1} \cdot R(\widetilde{J}_{\pm}\widetilde{X}, \widetilde{Y}) + f_n \cdot R(\widetilde{X}, \widetilde{Y}), \tag{2.17}$$

where $(f_n)_{n\in\mathbb{N}}$ *is the Fibonacci sequence defined by* $f_{n+2} = f_{n+1} + f_n$*,* $f_1 = f_2 = 1$*.*

ii) Golden Riemannian structure on $(\widetilde{M}, \widetilde{g})$ induced by two Golden structures on M_1 and M_2

For any vector field $\widetilde{X} = (X, Y) \in T(M_1 \times M_2)$ we define a linear map $\widetilde{J}$ of tangent space $T(M_1 \times M_2)$ into itself as follows:

$$\widetilde{J}\widetilde{X} = (J_1X, J_2Y), \tag{2.18}$$

where J_1 and J_2 are two Golden structures defined on M_1 and M_2, respectively. It follows that:

$$\begin{aligned} \widetilde{J}^2\widetilde{X} &= \widetilde{J}(J_1X, J_2Y) = (J_1^2X, J_2^2Y) = \\ &= (J_1X + X, J_2Y + Y) = (J_1X, J_2Y) + (X, Y). \end{aligned} \tag{2.19}$$

Also from $g_i(J_iX_i, Y_i) = g_i(X_i, J_iY_i)$, $i \in \{1, 2\}$, we get $\widetilde{g}(\widetilde{J}\widetilde{X}, \widetilde{Y}) = \widetilde{g}(\widetilde{X}, \widetilde{J}\widetilde{Y})$. Therefore, we can state the following result:

Theorem 2.8. *If* (M_1, g_1, J_1) *and* (M_2, g_2, J_2) *are Golden Riemannian manifolds, then there exists a Golden Riemannian structure* $\widetilde{J}$ *on* $(\widetilde{M}, \widetilde{g})$ *given by:*

$$\widetilde{J}\widetilde{X} = (J_1X, J_2Y), \tag{2.20}$$

for any $\widetilde{X} = (X, Y) \in T(\widetilde{M})$*, where* $\widetilde{M} = M_1 \times_f M_2$ *and* $\widetilde{g}(\widetilde{X}, \widetilde{Y}) = g_1(X_1, Y_1) + (f \circ p_1)^2 g_2(X_2, Y_2)$*, for any* $\widetilde{X} = (X_1, X_2), \widetilde{Y} = (Y_1, Y_2) \in T(\widetilde{M}) = T(M_1 \times_f M_2)$*.*

For the case of product Riemannian manifolds, from (2.5) we have:

Proposition 2.9. *Let* $(\widetilde{M} = M_1 \times M_2, \widetilde{g}, \widetilde{J})$ *(with* $\widetilde{g}$ *given by* (2.1) *for* $f = 1$ *and* $\widetilde{J}$ *given by* (2.18)*) be the product of the locally Golden Riemannian manifolds* (M_1, g_1, J_1) *and* (M_2, g_2, J_2)*. Then for any* $\widetilde{X}, \widetilde{Y}, \widetilde{Z} \in T(\widetilde{M}) = T(M_1 \times M_2)$*, the Riemannian curvature tensor verifies the relations:*

$$R(\widetilde{X}, \widetilde{Y})\widetilde{J}\widetilde{Z} = \widetilde{J}(R(\widetilde{X}, \widetilde{Y})\widetilde{Z}), \tag{2.21}$$

$$R(\widetilde{J}\widetilde{X}, \widetilde{Y}) = R(\widetilde{X}, \widetilde{J}\widetilde{Y}), \tag{2.22}$$

$$R(\widetilde{J}\widetilde{X}, \widetilde{J}\widetilde{Y}) = R(\widetilde{J}\widetilde{X}, \widetilde{Y}) + R(\widetilde{X}, \widetilde{Y}), \tag{2.23}$$

$$R(\widetilde{J}^{n+1}\widetilde{X}, \widetilde{Y}) = f_{n+1} \cdot R(\widetilde{J}\widetilde{X}, \widetilde{Y}) + f_n \cdot R(\widetilde{X}, \widetilde{Y}), \tag{2.24}$$

where $(f_n)_{n \in \mathbb{N}}$ *is the Fibonacci sequence defined by* $f_{n+2} = f_{n+1} + f_n$*,* $f_1 = f_2 = 1$*.*

Now we shall obtain a characterization of the Golden structure on the product of two Golden manifolds (M_1, J_1) and (M_2, J_2) in terms of *Golden maps*, that are smooth maps $\Phi : M_1 \to M_2$ satisfying:

$$T\Phi \circ J_1 = J_2 \circ T\Phi.$$

Remark that for the Golden structure $\widetilde{J} := (J_1, J_2)$ given by (2.18), the projections p_1 and p_2 on the two factors M_1 and M_2 are Golden maps. Indeed:

$$(Tp_i \circ \widetilde{J})(X_1, X_2) = Tp_i(J_1X_1, J_2X_2) = J_iX_i = J_i(Tp_i(X_1, X_2)),$$

$i \in \{1, 2\}$, for any $X_1 \in T(M_1)$ and $X_2 \in T(M_2)$.

Conversely, if we assume that the two projections p_1 and p_2 are Golden maps, then $\widetilde{J} = (J_1, J_2)$, since:

$$(Tp_i \circ \widetilde{J})(X_1, X_2) = J_i(Tp_i(X_1, X_2)) = J_iX_i,$$

$i \in \{1, 2\}$, for any $X_1 \in T(M_1)$ and $X_2 \in T(M_2)$, if we denote by $(Y_1, Y_2) =: \widetilde{J}(X_1, X_2)$, then $Y_i = J_iX_i$, $i \in \{1, 2\}$.

Therefore:

Proposition 2.10. *The Golden structure* $\widetilde{J} := (J_1, J_2)$ *given by* (2.18) *is the only Golden structure on the product manifold* $\widetilde{M} = M_1 \times M_2$ *such that the projections* p_1 *and* p_2 *on the two factors* M_1 *and* M_2 *are Golden maps.*

A necessary and sufficient condition for the warped product of two locally Golden Riemannian manifolds to be locally Golden will be further provided:

Theorem 2.11. *Let* $(\widetilde{M} = M_1 \times_f M_2, \widetilde{g}, \widetilde{J})$ *(with* $\widetilde{g}$ *given by* (2.1) *and* $\widetilde{J}$ *given by* (2.18)*) be the warped product of the locally Golden Riemannian manifolds* (M_1, g_1, J_1) *and* (M_2, g_2, J_2)*. Then* $(\widetilde{M} = M_1 \times_f M_2, \widetilde{g}, \widetilde{J})$ *is locally Golden if and only if:*

$$\begin{cases} (df^2 \circ J_1) \otimes I = df^2 \otimes J_2 \\ g_2(J_1 \cdot, \cdot) \cdot grad(f^2) = g_2(\cdot, \cdot) \cdot J_1(grad(f^2)) \end{cases}.$$

Proof. Replacing the expression of $\widetilde{\nabla}$ from (2.4), for any $X_1, Y_1 \in T(M_1)$ and $X_2, Y_2 \in T(M_2)$, we have:

$$(\widetilde{\nabla}_{(X_1,X_2)}\widetilde{J})(Y_1, Y_2) := \widetilde{\nabla}_{(X_1,X_2)}\widetilde{J}(Y_1, Y_2) - \widetilde{J}(\widetilde{\nabla}_{(X_1,X_2)}(Y_1, Y_2)) =$$

$$= ((^{M_1}\nabla_{X_1} J_1)Y_1 - \frac{1}{2} g_2(J_2X_2, Y_2) \cdot grad(f^2) + \frac{1}{2} g_2(X_2, Y_2) \cdot J_1(grad(f^2)),$$

$$(^{M_2}\nabla_{X_2} J_2)Y_2 + \frac{1}{2f^2}(J_1Y_1)(f^2)X_2 - \frac{1}{2f^2} Y_1(f^2) J_2X_2).$$

Under the assumptions $^{M_1}\nabla J_1 = 0$ and $^{M_2}\nabla J_2 = 0$ we get:

$$(\widetilde{\nabla}_{(X_1,X_2)}\widetilde{J})(Y_1, Y_2) = (-\frac{1}{2} g_2(J_2X_2, Y_2) \cdot grad(f^2) + \frac{1}{2} g_2(X_2, Y_2) \cdot J_1(grad(f^2)),$$

$$\frac{1}{2f^2}(J_1Y_1)(f^2)X_2 - \frac{1}{2f^2} Y_1(f^2) J_2X_2) =$$

$$= (-\frac{1}{2}[g_2(J_2X_2, Y_2) \cdot grad(f^2) - g_2(X_2, Y_2) \cdot J_1(grad(f^2))],$$

$$\frac{1}{2f^2}[df^2(J_1Y_1)X_2 - df^2(Y_1)J_2X_2]),$$

from where we obtain the conclusion. □

Theorem 2.12. *Let* $(\widetilde{M} = M_1 \times_f M_2, \widetilde{g}, \widetilde{J})$ *(with* $\widetilde{g}$ *given by* (2.1) *and* $\widetilde{J}$ *given by* (2.18)*) be the warped product of the Golden Riemannian manifolds* (M_1, g_1, J_1) *and* (M_2, g_2, J_2)*. If* M_1 *and* M_2 *have* J_1*- and* J_2*-invariant Ricci tensors, respectively (i.e.* $Q_{M_i} \circ J_i = J_i \circ Q_{M_i}$*,* $i \in \{1, 2\}$*), then* $\widetilde{M}$ *has* $\widetilde{J}$*-invariant Ricci tensor if and only if*

$$Hess(f)(J_1 \cdot, \cdot) - Hess(f)(\cdot, J_1 \cdot) \in \{0\} \times T(M_2).$$

Proof. If we denote by S, S_{M_1}, S_{M_2} the Ricci curvature tensors on $\widetilde{M}$, M_1 and M_2 and $\widetilde{S_{M_1}}$, $\widetilde{S_{M_2}}$ the lift on $\widetilde{M}$ of S_{M_1} and S_{M_2}, remark that the J_i-invariance of the Ricci tensor Q_{M_i}, $i \in \{1, 2\}$, is equivalent to $S_{M_i}(J_iX, Y) = S_{M_i}(X, J_iY)$, $i \in \{1, 2\}$, which implies $\widetilde{S_{M_i}}(\widetilde{J}X, Y) = \widetilde{S_{M_i}}(X, \widetilde{J}Y)$, $i \in \{1, 2\}$.

Now using Lemma 2.4, we have for any X, $Y \in \mathcal{L}(M_1)$:

$$S(\widetilde{J}X, Y) = \widetilde{S_{M_1}}(\widetilde{J}X, Y) - \frac{m}{f}H^f(\widetilde{J}X, Y) = \widetilde{S_{M_1}}(X, \widetilde{J}Y) - \frac{m}{f}H^f(\widetilde{J}X, Y) =$$

$$= S(X, \widetilde{J}Y) + \frac{m}{f}H^f(X, \widetilde{J}Y) - \frac{m}{f}H^f(\widetilde{J}X, Y),$$

where H^f is the lift on $\widetilde{M}$ of $Hess(f)$.

Also from Lemma 2.4, for any V, $W \in \mathcal{L}(M_2)$ we similarly obtain:

$$S(\widetilde{J}V, W) = \widetilde{S_{M_2}}(\widetilde{J}V, W) - [f\Delta(f) + (m-1)|grad(f)|^2]g_2(J_2V, W) =$$

$$= \widetilde{S_{M_2}}(V, \widetilde{J}W) - [f\Delta(f) + (m-1)|grad(f)|^2]g_2(V, J_2W) = S(V, \widetilde{J}W).$$

□

3 Example of Golden warped product Riemannian manifold

Let M be a submanifold in $\mathbb{R}^{2n}$ with the local coordinates $(x_1, y_1, x_2, y_2, ..., x_n, y_n)$ given by:

$$x_i = u\cos\alpha_i, \qquad y_i = u\sin\alpha_i, \tag{3.1}$$

for $i \in \{1, ..., n\}$, where $u > 0$ and α_i denote arbitrary parameters.

Let $X_i := \frac{\partial}{\partial x_i}$ and $Y_i := \frac{\partial}{\partial y_i}$, $i \in \{1, 2, ..., n\}$. Denote by:

$$(X^1, Y^1, ..., X^k, Y^k, X^{k+1}, Y^{k+1}, ..., X^n, Y^n) := (X^i, Y^i, X^j, Y^j),$$

for $i \in \{1, ..., k\}$, $j \in \{k+1, ..., n\}$, $k \in \{2, ..., n-1\}$, and consider the $(1,1)$-the tensor field $J : \Gamma(T\mathbb{R}^{2n}) \to \Gamma(T\mathbb{R}^{2n})$ defined by:

$$J(X^i, Y^i, X^j, Y^j) := (\sigma X^i, \sigma Y^i, \overline{\sigma}X^j, \overline{\sigma}Y^j), \tag{3.2}$$

where $\sigma = \frac{1+\sqrt{5}}{2}$ is the Golden number and $\overline{\sigma} = \frac{1-\sqrt{5}}{2} = 1 - \sigma$.

We can verify that J is a Golden structure on $\mathbb{R}^{2n}$ (i.e. $J^2 = J + I$) and for any (X^i, Y^i, X^j, Y^j), $(X'^i, Y'^i, X'^j, Y'^j) \in \Gamma(T\mathbb{R}^{2n})$, the scalar product $\langle\cdot,\cdot\rangle$ on $\mathbb{R}^{2n}$ is J-compatible:

$$\langle J(X^i, Y^i, X^j, Y^j), (X'^i, Y'^i, X'^j, Y'^j)\rangle = \langle(X^i, Y^i, X^j, Y^j), J(X'^i, Y'^i, X'^j, Y'^j)\rangle.$$

Therefore, $(\mathbb{R}^{2n}, \langle\cdot,\cdot\rangle, J)$ is a Golden Riemannian manifold.

Using a similar construction as in ([1]), we can check that the tangent bundle of M is spanned by the vectors:

$$Z_0 = \sum_{i=1}^{n} \left(\cos \alpha_i \frac{\partial}{\partial x_i} + \sin \alpha_i \frac{\partial}{\partial y_i} \right) \tag{3.3}$$

and

$$Z_i = -u \sin \alpha_i \frac{\partial}{\partial x_i} + u \cos \alpha_i \frac{\partial}{\partial y_i}, \tag{3.4}$$

for $i \in \{1, ..., n\}$. Then:

$$\|Z_0\|^2 = n, \quad \|Z_i\|^2 = u^2,\ i \in \{1, ..., n\} \tag{3.5}$$

and

$$Z_0 \perp Z_i, \quad Z_i \perp Z_j,\ i \neq j, \quad i, j \in \{1, ..., n\}. \tag{3.6}$$

From (3.2) and (3.3) we obtain:

$$JZ_0 = \sigma \sum_{i=1}^{k} \left(\cos \alpha_i \frac{\partial}{\partial x_i} + \sin \alpha_i \frac{\partial}{\partial y_i} \right) + \overline{\sigma} \sum_{j=k+1}^{n} \left(\cos \alpha_j \frac{\partial}{\partial x_j} + \sin \alpha_j \frac{\partial}{\partial y_j} \right) \tag{3.7}$$

and

$$JZ_i = \sigma Z_i, \quad JZ_j = \bar{\sigma} Z_j, \tag{3.8}$$

for $i \in \{1, ..., k\}$, $j \in \{k+1, ..., n\}$, $k \in \{2, ..., n-1\}$.

The Riemannian metric on M is given by:

$$g = n du^2 + u^2 \sum_{i=1}^{n} d\alpha_i^2 =: g_1 \times_u g_2. \tag{3.9}$$

Thus M is an $(n+1)$-dimensional warped product submanifold of the Golden Riemannian manifold $(\mathbb{R}^{2n}, \langle \cdot, \cdot \rangle, J)$ with the warping function u.

References

[1] M. Atceken, Warped Product Semi-Invariant Submanifolds in locally decomposable Riemannian manifolds, *Hacet. J. Math. Stat.*, **40**, no. 3, (2011), 401–407.

[2] M. Atceken and S. Keles, On the product Riemannian manifolds, *Differ. Geom. Dyn. Syst.*, **5**, no.1, (2003), 1–8.

[3] Y. B. Baik, A certain polynomial structure, *J. Korean Math. Soc.*, **16(80)**, no. 2, (1979), 167–175.

[4] R. L. Bishop and B. O'Neill, Manifolds of negative curvature, *Trans. Amer. Math. Soc.*, **145**, (1969), 1–49.

[5] M. Crasmareanu and C. E. Hretcanu, Golden differential geometry, *Chaos Solitons Fractals*, **38(5)**, (2008), 1229–1238.

[6] M. Crasmareanu, C. E. Hretcanu and M. I. Munteanu, Golden- and product-shaped hypersurfaces in real space forms, *Int. J. Geom. Methods Mod. Phys.*, **10(4)**, (2013), paper 1320006, 9 pp.

[7] S. I. Goldberg and K. Yano, Polynomial structures on manifolds, *Kodai Math. Sem. Rep.*, **22**, (1970), 199–218.

[8] S. I. Goldberg and N. C. Petridis, Differentiable solutions of algebraic equations on manifolds, *Kodai Math. Sem. Rep.*, **25**, (1973), 111–128.

[9] A. N. Hatzinikitas, A note on doubly warped product spaces, arXiv:1403.0204v1.2014.

[10] C. E. Hretcanu and M. C. Crasmareanu, On some invariant submanifolds in Riemannian manifold with Golden Structure, *An. Ştiinţ. Univ. Al. I. Cuza Iaşi. Mat. (N.S.)*, **53**, (2007), Suppl., 199–211.

[11] C. E. Hretcanu and M. C. Crasmareanu, Applications of the Golden Ratio on Riemannian Manifolds, *Turkish J. Math.*, **33(2)**, (2009), 179–191.

[12] C. E. Hretcanu and M. C. Crasmareanu, Metallic structures on Riemannian manifolds, *Rev. Un. Mat. Argentina*, **54(2)**, (2013), 15–27.

[13] W. J. Lu, f-Harmonic maps of doubly warped product manifolds, *Appl. Math. J. Chinese Univ.*, **28**, (2013), 240–252.

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