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On Some Inequalities with Operators in Hilbert Spaces

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Abstract: Operators defined on Hilbert spaces represent a major subfield of (or base for) the Functional Analysis. Several types of inequalities among such operators were established and studied in the last decades, mainly by the early '50s and then from the '80s. In this paper there are reviewed some of the most important types of inequalities, introduced and studied by H. Bohr, E. Heinz - T. Kato, T. Ando, etc. They were extended and/or sharpened by other authors, mentioned in the Introduction. Some of the definitions and proofs, found in several references, are completed (by the author) with specific formulas involving H-space operators, several details are also added to certain proofs and definitions as well. The main ways for establishing inequalities with operators are pointed out: scalar inequalities like the Cauchy-Schwarz and Bohr's inequalities over the complex field or on an H-space, certain identities with H-space operators, etc.

Keywords: Operators on Hilbert spaces, Bohr's inequality, Heinz's and Kato's inequalities.

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1 Introduction

Many and various types of inequalities, involving operators defined on Hilbert spaces, were introduced and studied in the latest 6-7 decades. The main class of H-space operators taken into account in such studies is the one of (bounded) selfadjoint operators on complex Hilbert spaces. A large number of papers approaching this subject of inequalities with operators were published, mainly in the '80s, and the names of a few most significant authors deserve mention: F. Kittaneh (1986, 1988), F. Kittaneh (1988), O. Hirzallah (2003), several Croatian authors headed by Academician J. Pečarić of Zagreb, F. Zhang (2007). But – as mentioned in the Abstract – several forerunning papers appeared around 1950. In the next Section 2, there are recalled basic definitions and notations from the theory of Hilbert space operators. The subsequent Section 3 is dedicated to operator inequalities induced by scalar inequalities over Hilbert spaces like the mixed Schwarz inequality, H. Bohr inequality, F. Heinz – T. Kato inequality, inequalities induced by operator positivity. The author of this paper identifies some ways to derive inequalities with H-Space operators, some relations between specific types of inequalities are discussed and several details are added to certain definitions and proofs in ten Remarks and four Propositions.

2 Preliminaries on Hilbert space operators

2.1 Basic definitions and notations

A Hilbert space over the complex field is denoted, in [6], as $\boldsymbol{H}$.The standard definition of a *norm* on an inner product space, including on a Hilbert space, is

$$(\forall\, x \in \boldsymbol{H}\,)\ \ ||\, x\, ||\ = \sqrt{\langle\, x\,,\ x\,\rangle}\,. \tag{2.1}$$

A linear operator $T : \boldsymbol{H} \longrightarrow \boldsymbol{H}$ obeys the natural property of linearity,

$$(\,\forall\, \alpha_1,\, \alpha_2 \in \mathbb{C}\ )\,(\,\forall\ x_1,\, x_2 \in \boldsymbol{H}\,)\quad T\,(\alpha_1\, x_1 + \alpha_2\, x_2) = \alpha_1\, T\, x_1 +\ \alpha_2\, T x_2\,. \tag{2.2}$$

Definition 2.1. *Domain* and *range* of an operator $T : \boldsymbol{H} \to \boldsymbol{H}$. This notation means that T is defined everywhere on $\boldsymbol{H}$, but it is possible that the domanin of T is only a subset (subspace) of $H : \mathfrak{D}_T \subseteq \boldsymbol{H}$. The range of T is defined by

$$\mathfrak{R}_T = \{\ y \in \boldsymbol{H} : (\ \exists\ \ x \in \boldsymbol{H}\ )\ \ y = T\, x\}\,. \tag{2.3}$$

The *kernel* and the *image* of T are (respectively) defined by

$$\text{Ker}\ T = \{x \in \boldsymbol{H} : T\, x = \mathbf{0}\}\quad \text{Im}\ T = \{y \in \boldsymbol{H} : (\,\exists\, x \in \mathfrak{D}_T)\ \ y = T\, x\}. \tag{2.4}$$

Remark 2.1. The image of T, as defined in (2.5), is slightly more general than its range of (2.4) since a posibly smaller domain (than the whole space $\boldsymbol{H}$) is considered. They coincide if $\mathfrak{D}_T = \boldsymbol{H}$. The zero vector is denoted as $\mathbf{0}$ for avoiding possible confusions with the zero scalar and/or the zero operator as well, which is denoted as 0 in most references. The two subspaces that occur in (2.5) can be also defined in an equivalent (may be simpler) way as in our textbook [3] :

$$\text{Ker}\ T = T_{-1}\ (\mathbf{0})\,,\quad \text{Im}\ T = T\,\boldsymbol{H}\quad \text{or}\quad \text{Im}\ T = T\,\mathfrak{D}_T. \tag{2.5}$$

In (2.6) , T_{-1} denotes the counterimage mapping, from $\mathfrak{R}_T$ to $\mathcal{P}(\mathfrak{D}_T)$: if $y \in \mathfrak{R}_T$ then its counterimage is $T_{-1}(y) = \{x \in \boldsymbol{H} : Tx = y\}$. It is well known that T is injective $\Leftrightarrow$ Ker $T = \{\mathbf{0}\}$ and it is surjective (or onto) $\Leftrightarrow$ Im $T = \boldsymbol{H}$. The identity operator $I_{\boldsymbol{H}}$ is (most often) denoted as $\mathbf{1}$. The connection between this $\mathbf{1}$, the zero operator 0 and the two subspaces that occur in (2.5) is

$$\text{Ker}\ \mathbf{1} = \{\mathbf{0}\}\,, \quad \text{Im}\ \mathbf{1}\ = \boldsymbol{H}\ , \quad \text{Ker}\ 0 = \boldsymbol{H}\ , \quad \text{Im}\ 0\ = \{\mathbf{0}\}.$$

The linear operations with operators are defined as usually : the sum $T_1 + T_2$ and $\alpha T =$ the multiplication by a scalar. The composition (or product) of two operators T_1, T_2 from $\boldsymbol{H}$ to $\boldsymbol{H}$ is defined, as usually for the composition of two maps on the same set / space :

$$(\ \forall\ x \in \boldsymbol{H}\ )\ \ (T_1\ T_2)\, x = T_2\ (T_1)\, x \underset{\text{not}}{=} (T_1 \circ T_2)\, x\,. \tag{2.6}$$

Endowed with these three operations, the set of operators defined on the Hilbert space $\boldsymbol{H}$ has the structure of a $C^{\star}-$ algebra, that is an involutive Banach algebra.

2.2 Classes of H-space operators and characterizations

Several inequalities involving operators defined on a Hilbert space $\boldsymbol{H}$ were formulated and studied for operators of certain (particular) types. Certain classes of H-space operators need a few preliminary definitions for being introduced. The first of them follows, but it implies the notion of a bounded operator, which follows to be presented together with other types after this one. According to [6], an operator $T : \boldsymbol{H} \longrightarrow \boldsymbol{H}$ is said to be bounded if there exists a positive number α such that $(\forall\ x \in \boldsymbol{H}\ )\ \ ||\, T\, x\, ||\ \leq \alpha\, ||\, x\, ||$. This class of bounded operators is denoted (in almost all references) as $\mathcal{B}(\boldsymbol{H})$. A large variety of classes (or types) of H-space operators can be found in [6]. A selection of five classes is presented in Jan Hamhalter's extended LN [7], together with characterizations for three of them. But some of them involve the notion of the adjoint of an operator $T : \boldsymbol{H} \longrightarrow \boldsymbol{H}$, and also the notion of norm. Two definitions of the *norm* of an H-space operator T are :

$$||\, T\, ||\ = \sup\{||\, T\, x\, ||\, /\, ||\, x\, ||\ : x \neq \mathbf{0}\} \text{ or } ||\, T\, ||\ = \sup_{||\, x\, || = ||\, y\, || = 1} |\, \langle\, y\,,\ T\, x\, \rangle\, |\,. \tag{2.7}$$

The above definition of (2.8) occurs in E. Heinz's paper [9] while the latter was given by Ch. Remling in [17]. According to [5], these two definitions are equivalent if T is selfadjoint.

Definition 2.2. ([6] or [17])The *adjoint* of an operator $T : \boldsymbol{H} \longrightarrow \boldsymbol{H}$, $T \in \mathcal{B}(\boldsymbol{H})$, denoted T^*, is defined by

$$(\forall\ x\ ,\ y \in \boldsymbol{H}\ )\ \ \langle\, T^* y\,,\ x\,\rangle = \langle\, y\,, T\, x\,\rangle. \tag{2.8}$$

The main properties of the adjoint operator were stated in Ch. Remling's just quoted lecture note :

Theorem 2.1 (Th. 6.1 in [20]). *Let* $S, T \in \mathcal{B}(\boldsymbol{H})$ *and* $c \in \mathbb{C}$. *Then* (a) $T^* \in \mathcal{B}(\boldsymbol{H})$; (b) $(S+T)^* = S^* + T^*$, $(cT)^* = \bar{c}\, T^*$; (c) $(S\,T)^* = T^* S^*$; (d) $T^{**} = T$; (e) *If* T *is invertible, then* T^* *is also invertible and* $(T^*)^{-1} = (T^{-1})^*$; (f) $\| T \| = \| T^* \|$, $\| T\, T^* \| = \| T^* T \| = \| T \|^2$ *(the* $C^{\star}-$ *property).*

Definition 2.3. Let $T \in \mathcal{B}(\boldsymbol{H})$. Then T is said to be

(i) *normal* *if* $T\,T^* = T^* T$;
(ii) *selfadjoint* *if* $T = T^*$;
(iii) *positive* *if* $\langle\, T\, x\, ,\, x\, \rangle \geq 0$ for all $x \in \boldsymbol{H}$
(iv) *unitary* *if* $T\,T^* = T^* T = \mathbf{1}$;
(v) *projection* *if* $T\,T^* = T^2 = T^* T$.

Characterizations of normal, selfadjoint and unitary operators are also presented in [7].

3 Inequalities with H-space operators induced by inequalities on $\mathbb{C}$ / H

3.1 Inequalities induced by operator positivity

If the adjoint operator T^* is considered, it is easy to see that *Def. 2.3* $-$ (iii) of a positive operator is equivalent to

$$\langle\, x\, ,\, T^* x\, \rangle \geq 0 \ \text{ for all } \ x \in \boldsymbol{H}\, . \tag{3.1}$$

Remark 3.1. The inequality $\langle\, T\, x\, ,\, x\, \rangle \geq 0$ or its equivalent of (3.1) make sense only if the respective inner products are real. This condition is often omitted, that is not explicitly stated. But part 2° of **1.3. Proposition** in [7] states that an inner product $\langle\, T\, x\, ,\, x\, \rangle \in \mathbb{R}$ if and only if T is selfadjoint.

In what follows, there are implied two or more H-space operators and we are going to denote them as $A,\ B,\ C, \ldots$ instead of T . Replacing, in the definition of a positive operator, T by the difference of the operators A , B , the next definition naturally follows.

Definition 3.1. If $A, B \in \mathcal{B}(\boldsymbol{H})$, then

$$A \geq B \underset{\text{def}}{\Longleftrightarrow} (\forall\, x \in \boldsymbol{H}\,) \quad \langle (\, A - B\,)\, x\, ,\, x\, \rangle \geq 0\, . \tag{3.2}$$

The positivity of a bounded operator A and the binary relation $A \geq B$ were also considered by Fuzhen Zhang in [18], with the mention that both operators should be self-adjoint. This definition implies a partial ordering over the class of bounded (selfadjoint) operators on an H-space. We state this assertion as

Proposition 3.1. *The relation defined by (3.2) is a partial ordering on* $\mathcal{B}(\boldsymbol{H})$.

Proof. (R) − Reflexivity : $A \geq A$ since

$$(\,\forall\, x \in \boldsymbol{H}\,)\ \ \langle (\,A - A\,)\, x\,,\, x\,\rangle = \langle 0\, x\,,\, x\,\rangle = \langle\, 0\,,\, x\,\rangle = 0\ . \tag{3.3}$$

The last equation in (3.3) is obvious for any inner product, and it is slightly weaker than the first axiom for any IP. See a *Note* next to (2.1). Hence $A - A = 0 \Longrightarrow A = A$, the equality case of $A \geq A$.(AS) − Antisymmetry : $A \geq B$ & $B \geq A \Longrightarrow A = B$. Indeed, according to *Def. 2.2* − *(iii)* , the two inequalities in the left side of this implication mean that

$$(\forall\, x \in \boldsymbol{H})\ \langle (A-B)x, x\rangle \geq 0\ \&\ \langle (B-A)x, x\rangle = -\langle (A-B)x, x\rangle \geq 0 \Longrightarrow$$

$$[\langle (A-B)x, x\rangle \geq 0\ \langle (A-B)x, x\rangle \leq 0] \Longrightarrow \langle (A-B)x, x\rangle = 0 \ \Longrightarrow\ A = B\ .$$

(Tr) − Transitivity : $A \geq B$ & $B \geq C \Longrightarrow A \geq C$. In a similar way,

$$A \geq B\ \&\ B \geq C \quad\Longrightarrow\quad [\langle (A-B)x, x\rangle \geq 0\ \ \&\ \ \langle (B-C)x, x\rangle \geq 0] \quad\Longrightarrow$$

$$(\forall\; x \in H\,)\quad \langle (\,A - C\,)\; x\,,\, x\,\rangle \geq 0 \quad\Longrightarrow\quad A \geq C\,.$$

The but last inequality obviously follows by adding − side-by-side − the two inequalities inside [...]. □

Remark 3.2. In view of the earlier *Remark 3.1*, the statement of this *Proposition 3.1* would have had to be restricted to the (subspace / subalgebra of) selfadjoint operators since the order relation of (3.2) holds for this class of operators only. However, we have just noticed that, in most publications dealing with positive operators, this restriction is not explicitly stated. In fact, the operator differences that occur in (3.2) and in the subsequent equations should be selfadjoint, and not necessarily the operators A, B, C themselves. As a matter of terminology, the self-adjoint operators (or matrices) are equivalently termed *Hermitian* in Serge Lang's volume Linear Algebra (Springer-Verlag, 1987) − [13].

3.2 T. Kato's inequality involving the domains and the norms of images

Another binary relation between two operators was defined by T. Kato in an article of 1952.

Definition 3.2. ([10]) Let S ,T be two linear operators. Then

$$S \ll T \;\Leftrightarrow [\mathfrak{D}_S \supseteq \mathfrak{D}_T \quad \& \quad (\forall\, x \in \mathfrak{D}_T)\;\; ||\, S\,x\,|| \;\leq ||\,T\,x\,||\,]\,. \tag{3.4}$$

Proposition 3.2. *The relation defined by (3.10) is a partial ordering on the algebra of linear operators defined on* $\boldsymbol{H}$.

Proof. The inclusion between the domains of S and T is a simple subset inclusion over the parts of $\boldsymbol{H}$ and it clearly satisfies the three properties of a partial ordering. As regards the inequality between the norms of the two images, let us recall that the norm of an element (vector) in a H-space is defined as in (2.2) : $||\,x\,|| \;=\; \sqrt{\langle\, x\,, x\,\rangle}$. Hence

$$||\,S\,x\,|| \;=\; \sqrt{\langle\, S\,x\,,\; S\,x\,\rangle} \quad \text{and} \quad ||\,T\,x\,|| \;=\; \sqrt{\langle\, T\,x\,,\; T\,x\,\rangle}. \tag{3.5}$$

It follows that $(\forall\, x \in \mathfrak{D}_T)\;\; ||\,S\,x\,|| \leq ||\,T\,x\,|| \Leftrightarrow \sqrt{\langle\, S\,x\,,\; S\,x\,\rangle} \leq \sqrt{\langle\, T\,x\,,\; T\,x\,\rangle} \Leftrightarrow$

$$\Leftrightarrow \langle\, S\,x\,,\; S\,x\,\rangle \leq \langle\, T\,x\,,\; T\,x\,\rangle \Leftrightarrow \langle\,(T-S\,)\,x\,,\; (T-S\,)\,x\,\rangle \geq 0\,. \tag{3.6}$$

The inequality in (3.10) obviously satisfies the three axioms of a partial order (R) : $||\; S\,x\;|| \;\;\leq ||\; S\,x\;||$; (AS) : $||\,S\,x\,|| \;\leq ||\,T\,x\,||$ and $||\,T\,x\,|| \;\leq ||\,S\,x\,|| \Longrightarrow$, by (3.5) , $T - S = 0 \Longrightarrow S = T$. The transitivity holds, too. □

In *Definition 3.2* and in the statement of this Proposition there are not assumed any properties of the operators thereof. However, the author asserts, for the case when S and T are selfadjoint, that $S \ll T$ is equivalent to $S^{\,2} \ll T^{\,2}$ in the sense of F. Rellich [16]. More generally, if S and T are closed and have everywhere closed domains, then

$$S \ll (\; S^*S\;)^{1/2} \ll S \quad \text{and} \quad T \ll (\; T^*T\;)^{1/2} \ll T$$

so that $S \ll T$ is equivalent to

$$(\; S^*S\;)^{1/2} \ll (\; T^*T\;)^{1/2} \;\text{, i.e. to}\;\; S^*S \ll T^*T\;. \tag{3.7}$$

We close these quotations from T. Kato's article [11] by a Corollary to this author's Theorem 1. Let A and B be selfadjoint operators and let $A \ll B$. If $A^{\,-1}$ exists, then $B^{\,-1}$ also exists and $B^{\,-1} \leq A^{\,-1}$.

Remark 3.3. Let us see that the operator inequalities defined in (3.2) and (3.4) do not effectively follow from some scalar inequalities, but they are defined in terms of the scalar product and norms on the images of the two operators, that hold in the H-space $\boldsymbol{H}$. The order relation between two operators $A,\ B \in \mathcal{B}(\boldsymbol{H})$ was also defined, in an equivalent way, by S.S. Dragomir as follows.

Definition 3.3. ([5]) If $A,\ B$ are selfadjoint operators on $\boldsymbol{H}$ then

$$A \leq B \Leftrightarrow \langle\, A\, x\,,\, x\,\rangle \leq \langle\, B\, x\,,\, x\,\rangle \quad \text{for all} \quad x \in \boldsymbol{H}\ . \tag{3.8}$$

It is obvious that the definition in (3.8) is equivalent to the one in (3.2). It simply suffices to interchange $A,\ B$ and to see that $\langle\, A\, x,\ x\,\rangle \leq \langle\, B\, x,\ x\,\rangle \Leftrightarrow \langle (B-A)\, x\,, x\,\rangle \geq 0$. This author's *Theorem 2*, next to this definition, states that the relation $A \leq B$ is reflexive, transitive and antisymmetric − the properties proved in our earlier *Proposition 3.1.* The statement of S.S. Dragomir's *Theorem 2* also includes two more properties that involve a third operator C and two real scalars, as well :

4. If $A \leq B$ and $\alpha \geq 0$ then

$$A + C \leq B + C \quad \text{and} \quad \alpha\, A \leq \alpha\, B, \quad -A \geq -B\ . \tag{3.9}$$

5. If

$$\alpha \leq \beta \quad \text{then} \quad \alpha\ A \leq \beta\ A\ . \tag{3.10}$$

Under similar assumptions on one / more operators, other two theorems are stated (without proofs) by S.S. Dragomir :

Theorem 3 ([5]). *Let A be a positive selfadjoint operator on H. Then*

$$||\, A\ x\, ||^2 \leq ||\ A\ ||^2\ \langle\, A\, x\,,\, x\,\rangle \quad \textit{for any} \quad x \in H\ . \tag{3.11}$$

Theorem 4 ([5]). *Let $A_n,\ B$ with $n \geq 1$ be positive selfadjoint operators on $\boldsymbol{H}$ with the property that*

$$A_1 \leq A_2 \leq\ \ldots\ A_n \leq\ \ldots\ B\ . \tag{3.12}$$

Then there exists a bounded selfadjoint operator A defined on $\boldsymbol{H}$ such that

$$A_n \leq A \leq B \quad \textit{for all} \ \ n \geq 1 \ \ \textit{and} \quad \lim_{n \to \infty} A_n\, x = A\, x\ . \tag{3.13}$$

Remark 3.4. It is clear then the operators A_n $(\, n \geq 1)$ in the statement of the latter Theorem are the terms of an increasing sequence of operators, upper-bounded by an operator B , with respect to the order relation of (3.2) / (3.8) . The operator A of (3.13) appears to be an upper margin of $\{A_n :\, n \geq 1\}$. This property is similar to a feature of increasing and upper-bounded real sequences in Calculus.

3.3 Operator inequalities induced by Cauchy-Schwarz type (scalar) inequalities

One of the most important inequalities over an inner product space (which is necessarily a normed space as well) is the Cauchy – Schwarz inequality. As a detail, some authors call it the Cauchy – Schwarz – Bunjakovski inequality. Obviously, it also holds on an H-space:

$$\text{(C-S-B)}\ (\forall\, x,\, y \in \boldsymbol{H})\ \ |\langle\, x,\, y\,\rangle|^2 \le \langle\, x,\, x\,\rangle\, \langle y,\, y\rangle \Leftrightarrow |\langle\, x,\, y\,\rangle| \le ||\, x\,||\, ||\, y\,||. \quad (3.14)$$

S.S. Dragomir presents a generalization of inequality (3.14). In each of the three inner products that occur in its first (left) version, one of its factors (which are vectors in $\boldsymbol{H}$) is replaced by its (their) image(s) through a selfadjoint and positive operator A:

$$(\forall x,\, y \in \boldsymbol{H})\ |\langle\, A\,x\,,\, y\,\rangle\,|^2 \le \langle\, A\,x\,,\, x\,\rangle\,\langle\, A\,y\,,\, y\,\rangle. \quad (3.15)$$

Remark 3.5. Let us see that the two IPs in the right side of (3.21) are positive, just because $A \ge 0$. The connection from (3.14) to (3.15) illustrates a way to obtain inequalities with operators, suggested by the title of this subsection. In fact, inequality (3.14) is a generalization of the Cauchy – Schwarz inequality since $[(3.15)\ \ \&\ \ A = \mathbf{1}] \Longrightarrow (3.14)$. The inequality (3.15) , with T instead of A , was presented in the Introduction to F. Kittaneh's article [12] and called the *Schwarz inequality for positive operators.*

Many generalizations of certain inequalities with H-space operators were formulated and studied by taking real (and not natural or 1/2) powers of operators there involved. It comes to formulas including – for example – powers of operators of the form A^{ν}, B^{ν} or $B^{1-\nu}$ with $\nu \in \mathbb{R}_+$. Fuad Kittaneh presents, in [12], such an inequality with real powers of the absolute values of T and its adjoint T^*. In fact, F.K. mentions that it was earlier established by T. Kato in his paper [11] of 1952, and he calls it the

Mixed Schwarz inequality : If $T \in \mathcal{B}(\boldsymbol{H})$, $x\,,\, y \in \boldsymbol{H}$ and $0 \le \alpha \le 1$ then

$$\langle\, T\,x\,,\ y\,\rangle\,|^2 \le \langle\,|\,T\,|^{2\,\alpha}\,x\,,\ x\,\rangle\,\langle\,|\,T^*\,|^{2(1-\alpha)}\,y\,,\ y\,\rangle\ . \quad (3.16)$$

Remark 3.6. In (3.16), $|\,T\,| = (\,T^*T\,)^{1/2}$ and $|\,T^*| = (\,T^*T\,)^{1/2}$. These two equalities, taken to (3.16) , lead to an equivalent expression of the above inequality :

$$\langle\, T\,x\,,\ y\,\rangle\,|^2 \le \langle\,(\,T^*T\,)^{\alpha}\,x\,,\ x\,\rangle\,\langle\,(T^*T\,)^{1-\alpha}\,y\,,\ y\,\rangle\ . \quad (3.17)$$

3.4 Bohr – type inequalities with H-space operators

In certain cases, an inequality involving one or two (bounded self-adjoint or only positive) operators like $T,\ A,\ B$ defined on a complex H-space $\boldsymbol{H}$, can be obtained from a "scalar" inequality which holds on $\mathbb{C}$ or in a Hilbert space by replacing one or two arguments like complex numbers $z,\, w \in \mathbb{C}$ or elements (vectors) $x\,,\ y \in \boldsymbol{H}$ by operators. Obviously, the expressions with operators thus obtained should result in H-space operators as well, based on operations with such operators like those in the algebra $\mathcal{B}(\boldsymbol{H})$. In a more general approach, new inequalities can be derived by use of certain operator functions of the form $f\,(A)$, with specific properties imposed to f.

For the former approach, it is claimed (in many references) that the classical Harald Bohr's inequality for scalars was presented in [2] – *Acta Mathematica* **45** (1924) – under the form :

$$|\, z + w\,|^2 \leq p\ |\, z\,|^2 + q\ |\, w\,|^2 \tag{3.18}$$

where $z\,, w \in \mathbb{C}$ & $p\,,\, q \in \mathbb{R}$ such that

$$p\,,\ q > 1 \ \text{ and } \ 1/p + 1/q = 1\ . \tag{3.19}$$

In several sources, the real numbers $p\,,\ q$ are called *conjugate exponents*. However, see the next *Remark* on the original form of H. Bohr's inequality.

This expression (3.18), with conditions (3.19), of H. Bohr's inequality over $\mathbb{C}$ is also presented in the article [4] by W-S. Cheung and J. Pečarić, where it is asserted that equality occurs in (3.18) if and only if $w = (\,p-1)\,z$. In [4] , the authors cite a generalization of the Bohr inequality (3.18), due to J. Pečarić & S.S. Dragomir [15], to the context of normed vector spaces. If $(V,\ ||\ \ ||\,)$ is such an NVS and the real $p\,, q$ satisfy (3.19) , then $||\, x + y\,||^2 \leq p\ ||\, x\,||^2 + q\ ||\, y\,||^2$ for any $x\,,\ y \in V.$

A variant of inequality (3.18) is presented, for instance, in O. Hirzallah's article [10] as well as in F. Zhang's [18] : if $a\,, b \in \mathbb{C}$ and $p\,, q \in \mathbb{R}$ also satisfy conditions (3.19) , then

$$|\, a - b\,|^2 \leq p\ |\, a\,|^2 + q\ |\, b\,|^2. \tag{3.20}$$

Obviously, equivalence between (3.18) and (3.20) follows by the replacements $z \leftrightarrows a$ & $w \leftrightarrows -b$.

Remark 3.7. In many references dedicated to Bohr type inequalities, it is asserted that expression (3.18) or (3.20) of the classical H. Bohr inequality was given in the (above quoted) reference [2]. In fact, that article was dedicated to almost periodic functions, to their Fourier expansions and to Dirichlet series. We did not find such an inequality in any of the 99 pages of that study. The only similar inequality appears at page 78 of [2]:

$$|\, a+b\,|^2 \le (1+c)\,|\, a\,|^2 + \left(1+\frac{1}{c}\right)\,|\, b\,|^2 \tag{3.21}$$

where $a, b \in \mathbb{C}$ and $c > 0$. This original form of the Bohr inequality is equally presented (and exactly quoted from [2]) in the paper [14] by J. Pečarić & Th. Rassias (*J.M.A.A.*, 1993), with other symbols for the two complex numbers: $a \to z_1$ & $b \to z_2$.

The latter variant (3.20) is better suited than (3.18) for stating operator inequalities of Bohr type. In [21] , the author quotes a result of [10] which generalizes (3.20) , namely O. Hirzallah's

Theorem 1 ([10]) *If* A , $B \in \mathcal{B}(\boldsymbol{H})$, $p\,, q > 1$ *with* $1/p + 1/q = 1$ *and* $p \le q$ *then*

$$|\, A - B\,|^2 + |\,(1-p)\,A - B\,|^2 \le p\;|\,A\,|^2 + q\;|\,B\,|^2. \tag{3.22}$$

The proof of inequality (3.22) is given in [10] at page 579. O.H. presents an operator version of inequality (3.20) in the *Corollary 1* of this *Theorem 1* .

Corollary 1 ([10]) *Let* A , $B \in \mathcal{B}(\boldsymbol{H})$, $p\,,\, q > 1$ *with* $1/p + 1/q = 1$. *Then*

$$|\, A - B\,|^2 \le p\;|\,A\,|^2 + q\;|\,B\,|^2. \tag{3.23}$$

with equality if and only if $(1-p)\;A = B$.

The absolute value of an operator (which occurs in the previous two equations) has been defined in our earlier *Remark 3.6.*

We present sketches of the proofs of these two results, followed by a Proposition which brings several details. The proof of the inequality (3.22) starts by the expansions of the first term in its left side:

$$|\, A - B\,|^2 = (\,A - B\,)^*(\,A - B\,) = (\,A^* - B^*)\,(\,A - B\,) = \,\ldots$$

$$\ldots = |\,A\,|^2 - (\,A^*B + B^*A\,) + |\,B\,|^2\;. \tag{3.24}$$

A similar expansion for the second term in the left hand of (3.22) is

$$|\,(1-p)\,A - B\,|^2 = [(1-p)\,A - B\,]^*\,[(1-p)\,A - B\,] = \,\ldots$$

$$\ldots = (1-p)^2\,|\,A\,|^2 + (\,p-1)\,(A\,B^* + A^*\,B\,) + |\,B\,|^2. \tag{3.25}$$

Summing up − side by side − equations (3.24) & (3.25) , it is obtained

$$|\,A - B\,|^2 + |\,(1-p)\,A - B\,|^2 = \,\ldots$$

$$\ldots = [1 + (1-p)^2]\,|\,A\,|^2 + 2\,|\,B\,|^2 - \,(2-p)\,(\,A^*B + B^*A\,) =$$

$$= (\,p^2 - 2\,p + 2)\,|\,A\,|^2 + 2\,|\,B\,|^2 + (\,p-2)\,(\,A^*B + B^*A\,)\,. \tag{3.26}$$

Next, the right side of (3.22), that is $p|\,A\,|^2{+}q|\,B\,|^2$, is subtracted from both (leftmost and rightmost) sides of (3.26), resulting in

$$(1-p)^2\,|\,A\,|^2 + |\,B\,|^2 - (\,2-p)\,(\,A^*B + B^*A\,) = \,\ldots = (\,p-2)\,(\,p-1)\,|\,A\,|^2 +$$

$$+[(\,p-2)\,/\,(\,p-1)]\;|\,B\,|^2 + (\,p-2)\,(\,A^*B + B^*A\,). \tag{3.27}$$

Obviously, $(\,p-2)$ can be taken out as a common factor from the rightmost side of (3.27), and it follows that

$$|\,A-B\,|^2 + |\,(1-p)\,A - B\,|^2 - (\,2-p)\,(\,A^*B + B^*A\,) =$$

$$= (\,p-2)\,\left[(\,p-1)\,|\,A\,|^2 + [1\,/\,(\,p-1)]\;|\,B\,|^2 + A^*B + B^*A\right]\,. \tag{3.28}$$

Other calculations lead to the conclusion that the right side of (3.28) is negative (non-positive), what implies the inequality (3.22) in the statement of this Theorem.

As regards the proof of *Corollary 1*, O. Hirzallah takes into account both possibilities for the conjugate exponents $p\,,\,q$. In fact, the author states that, if $p \le q$ then inequality (3.23) follows by *Theorem 1.* If the symmetric inequality $q \le p$ holds, then (3.28) becomes

$$|\,A-B\,|^2 + |\,(1-q)\,B - A\,|^2 \le p\;|\,A\,|^2 + q\;|\,B\,|^2. \tag{3.29}$$

In the proposition which follows, we add some details to O. Hirzallah's proofs of the previous *Theorem 1* and *Corollary 1*, a proof of (3.29), etc.

Proposition 3.3 (*Addenda to some earlier presented results and their proofs*)**.**

(i) Proof of the equality case in (3.23).

(ii) The operator version of the original H. Bohr's scalar inequality (3.21): If $a\,,b \in \mathbb{C}\,,\;\; c \in \mathbb{R}_+^*\;\;(c > 0\,)$ and $A\;,\;B \in \mathcal{B}(\boldsymbol{H})$ then

$$|\,A-B\,|^2 \le (1+c)\,|\,A\,|^2 + \left(1+\frac{1}{c}\right)\;|\,B\,|^2. \tag{3.30}$$

(iii) Addenda to the proofs of inequalities (3.22), (3.23) and (3.29).

Proofs. (i) As we have just earlier mentioned, it is stated – in the *Corollary 1* of [10] – that the equality in (3.23) occurs if and only if $(1-p)\,A = B$.

Let us see that the implication (3.22) $\Longrightarrow$ (3.23) is immediate if $B = (1-p)\,A$ is taken inside the second term of the left side of (3.22). Next, going back to the scalar inequality (3.18), it is (more or less) easy to see that it turns into an equality if $w = (p-1)\,z$. Indeed, $|\,z+w\,|^2 \longrightarrow$

$$\longrightarrow |\,z+(p-1)\,z\,|^2 = \;\ldots\; = (\,z+(p-1)\,z\,)\,(\,\overline{z}+\;(p-1)\overline{z}\,) =$$

$$= |\,z\,|^2 + 2\,(p-1)\,|\,z\,|^2 + (p-1)^2\,|\,z\,|^2 = p^2\,|\,z\,|^2. \tag{3.31}$$

The right side $p\,|\,z\,|^2 + q\,|\,w\,|^2$ of (3.18) becomes
$p\;|\,z\,|^2 + q\;|(\,p-1)\,z|^2 = p\;|\,z\,|^2 + q\;\,(p-1)^2\,|\,z\,|^2 = [\,p + q\,(\,p^2 - 2\,p + 1)]\,|\,z\,|^2 =$

$$= [\;p + p\;q\;(p-2) + q]\,|\,z\,|^2 = p\;q\;(1+p-2)\,|\,z\,|^2 = p\;q\;(p-1)\,|\,z\,|^2. \tag{3.32}$$

The second equality on line (3.32) follows from the defining equation for the conjugate exponents $p,\,q$ of (3.19) :

$$\frac{1}{p} + \frac{1}{q} = 1 \;\Longrightarrow\; \frac{p+q}{p\,q} = 1 \;\Longrightarrow\; p+q = p\,q\,. \tag{3.33}$$

Next, (3.33) $\Longrightarrow$ $(p-1)\;q = p$ which implies that the leftmost side of (3.31) is just equal to the rightmost side of (3.32).

As regards the earlier presented *Theorem 1* and *Corollary 1* by O. Hirzallah [10], Eq. (3.29) would follow from (3.18) by taking operators instead of complex numbers, namely $w \to A$ & $z \to -\,B$. However, we are going to address this issue in part (iii) of our Proposition.

(ii) We have presented the operator version of the original Bohr inequality in (3.30. It still follows to see the connection between the positive parameter c and the conjugate exponents $p,\,q$. A simple comparison between (3.29) and (3.30) leads to

$$1 + c = p \quad \& \quad 1 + \frac{1}{c} = q\,. \tag{3.34}$$

Expressions (3.34) $\Longrightarrow$ $p > 1,\; q > 1$ and $1\,/\,p + 1\,/\,q = 1\,/\,(1+c) + c\,/\,(c+1) = 1\,.$ Hence, the equivalence between (3.29) and (3.30) holds if we replace, in (3.21), $a \longrightarrow A$ and $b \longrightarrow -\,B$.

(iii) Some more details on the proof of O. Hirzallah's *Theorem 1* and *Corollary 1* of [10] follow. In the proof of (3.22), let us first see how the rightmost sides in (3.24) and (3.25) were obtained.

$|\,A - B\,|^2 + |\,(1-p)\,A - B\,|^2 =$ (using (3.24) & (3.25))
$= |\,A\,|^2 - (A^*B + B^*A) + |\,B\,|^2 + (1-p)^2\,|A|^2 - (1-p)\,(A^* + A^*B) + |B|^2 =$

$$= [1 + (1-p)^2]\,|\,A\,|^2 + 2\,|\,B\,|^2 - (2-\,p)\,(\,A^*B + B^*A\,) =$$

$$= (p^2 - 2p + 2)\,|\,A\,|^2 + 2\,|\,B\,|^2 + (p-2)\,(A^*B + B^*A)\,. \tag{3.35}$$

Subtracting $p\,|\,A\,|^2 + q\,|\,B\,|^2$ from both sides of (3.35) , it follows that

$$|\,A - B\,|^2 + |\,(1-p)\,A - B\,|^2 - (p\,|\,A\,|^2 + q\,|\,B\,|^2) =$$
$$= (p^2 - 2p + 2)\,|\,A\,|^2 + 2\,|\,B\,|^2 + (p-2)\,(A^*B + B^*A) - (p|\,A\,|^2 + q|\,B\,|^2) =$$

$$= (p^2 - 3p + 2)\,|\,A\,|^2 + (2-q)\,|\,B\,|^2 + (p-2)\,(A^*B + B^*A), \tag{3.36}$$

The first factor in the rightmost side of (3.36) can be written as $(p-2)(p-1)$ while

$$2 - q = 2 - p/(p-1) = (p-2)\,/\,(p-1),$$

as it follows from (3.33) $\Longrightarrow$ $(p-1)\;q = p$. Consequently, $(p-2)$ can be taken as a common factor from (33.36), what leads to inequality (3.29). This factor is a negative real number since the assumption $p \le q$ in the statement of Theorem 1 plus (3.19) $\Longrightarrow$

$$\Longrightarrow\; 1 = \frac{1}{p} + \frac{1}{q} \le \frac{1}{p} + \frac{1}{p} = \frac{2}{p} \;\Longrightarrow\; \frac{2}{p} \ge 1 \;\Longrightarrow\; p \le 2\,.$$

Finally, (3.28) with (3.27) would lead to the negativity of the leftmost side in (3.36) provided the sum of the three operators between [...] in the right side of (3.28) represents a positive operator. This is derived in [10] (but without a detailed proof) from

$$(p-1)\,|\,A\,|^2 + \left(\frac{1}{p-1}\right)|\,B\,|^2 + A^*B + B^*A = \left|\sqrt{p-1}\,A + \frac{1}{\sqrt{p-1}}\,B\right|^2. \tag{3.37}$$

The right side of (3.37) can be expanded as follows.

$$\left|\sqrt{p-1}\,A + \frac{1}{\sqrt{p-1}}\,B\right|^2 =$$
$$= \left[\sqrt{p-1}\,A^* + \frac{1}{\sqrt{p-1}}\,B^*\right]\,\left[\sqrt{p-1}\,A + \frac{1}{\sqrt{p-1}}\,B\right] =$$
$$= (p-1)\,|\,A\,|^2 + [1\,/\,(p-1)]\;|\,B\,|^2 + (A^*B + B^*A)\,.$$

The operator in the right side of (3.37) is (obviously) positive and, with its negative factor $(p-2)$ of (3.34), the inequality in (3.28) follows. As regards O. Hirzallah's proof for Corollary 1, in both alternatives on p, q, that is $p \le q$ and then $q \le p$, we do not bring any more details or comments on author's proof, but let us simply notice that (3.29) would follow from (3.28) if the operator $(1-p)^2 \,|\, A\,|^2 + |\, B\,|^2 \ge 0$. In the Remark which follows, we recall a somehow symmetric to (3.28) inequality, and we then continue with another approach to the study of Bohr-type inequalities with H-space operators, based on operator identities. □

Remark 3.8. Some additional results, other than O. Hirzallah's just presented Theorem 1 with its Corollary, were given in the article [4] by W.-S. Cheung and J. Pečarić. These authors recall Theorem 1 of [10] with its proof (earlier presented and completed), with the Remark 1 which states that their Theorem 1 – (i) is equivalent to (3.28) since $1 < p \le 2 \Longleftrightarrow 1 < p \le q$. Their assumptions on p, q are $1/p + 1/q = 1$ and $1 < p \le 2$. In view of a consequence of our earlier equation (3.29), namely $q = p/(p-1) \ge p > 1$, it follows that $q > 1$ is also satisfied. But it is more significant that another inequality is stated and proved in Theorem 1 – (ii) of [4],

$$|\, A - B\,|^2 + |\, A - (1-q)\, B\,|^2 \ge p\; |\, A\,|^2 + q\; |\, B\,|^2. \tag{3.38}$$

Obviously, it is practically equivalent to inequality (3.29) of [10], but O. Hirzallah's hypothesis $q \le p$ is no more assumed here. The proof is similar to that of (3.28). The expression of $|\, A - B\,|^2$ is just the same of (3.24). As regards the second term,

$$|\, A - (1-q)\, B\,|^2 = (1-q)\,|\, B\,|^2 + |\, A\,|^2 - (1-q)\,(A^*B + B^*A)\,. \tag{3.39}$$

Adding (again), side by side, equation (3.24) with (3.39) and subtracting $p\,|\,A|^2 + q\,|B|^2$ it is obtained

$$|\, A - B\,|^2 + |\, A - (1-q)\, B\,|^2 - p\; |\, A\,|^2 - q\; |\, B\,|^2 = \ldots$$

$$\ldots = (\, q^2 - 3\,q + 2)\,|\, B\,|^2 + (\, 2 - p\,)\,|\, A\,|^2 + (\, q-2\,)\,(\, A^*B + B^*A\,) =$$

$$= (\, q-2\,)\,(\, q-1\,)\,|\, B\,|^2 + \frac{q-2}{q-1}\,|\, A\,|^2 + (\, q-2\,)\,(\, A^*B + B^*A\,) =$$

$$= (\, q-2\,)\,\left[(\, q-1\,)\,|\, B\,|^2 + \frac{1}{q-1}\,|\, A\,|^2 + (\, A^*B + B^*A\,)\right].$$

Since $1 < p \le 2 \implies q \ge 2$ we have

$$|\, A - B\,|^2 + |\, A - (1-q)\, B\,|^2 - p\,|\, A\,|^2 - q\,|\, B\,|^2 = (\, q-2\,)\,\left|\sqrt{p-1}\,A + \frac{1}{\sqrt{p-1}}\,B\right|^2 \ge 0$$

and inequality (3.39) is thus proved. In the particular case when $p = 2 \Longrightarrow q = 2$, it follows from the previous inequalities that

$$|\,A-B\,|^2+|\,A+B\,|^2\leq 2\,|\,A\,|^2+2\,|\,B\,|^2\leq |\,A-B\,|^2+|\,A+B\,|^2 \implies$$

$$\implies |\,A-B\,|^2+|\,A+B\,|^2=2\,|\,A\,|^2+2\,|\,B\,|^2$$

which is the parallelogram law for H-space operators.

3.5 Bohr – type inequality via identities

In his article [18] of *J.M.A.A.* (2007), F. Zhang obtains some earlier presented Bohr type inequalities by use of operator identities and he also formulates certain extensions thereof, whose proofs involve 2-by-2 block matrices with operator entries. After quoting O. Hirzallah's Theorem 1 and Corollary 1, that is the inequalities (3.22) and (3.23), he states **Theorem 2**:

Let A, $B\in\mathcal{B}(\boldsymbol{H})$, $p,\ q>1$, $1/p+1/q=1$. *Then*

$$|\,A-B\,|^2+\left|\sqrt{p/q}\;A+\sqrt{q/p}\;B\,\right|^2=p\,|\,A\,|^2+q\,|\,B\,|^2. \qquad (3.40)$$

Equivalently, for any α , $0\leq\alpha\leq 1$,

$$|\,\alpha A+(1-\alpha)\,B\,|^2+\alpha\,(1-\alpha)\,|\,A-B\,|^2=\alpha\,|\,A\,|^2+(1-\alpha)\,|\,B\,|^2. \qquad (3.41)$$

The operator $|\,A-B\,|^{\,2}$ is expanded exactly like in (3.24), under the equivalent form

$$|\,A-B\,|^{\,2}=|\,A\,|^{\,2}+|\,B\,|^{\,2}-(\,A^*B+B^*A\,). \qquad (3.42)$$

Similarly, the next term in the left side of (3.40) becomes

$$\left|\sqrt{p/q}\;A+\sqrt{q/p}\;B\,\right|^2=(p/q)\,|\,A\,|^2+(q/p)\,|\,B\,|^2+(\,A^*B+B^*A\,). \qquad (3.43)$$

Obviously, (3.43) + (3.44) $\implies$

$$|\,A-B\,|^2+\left|\sqrt{p/q}\;A+\sqrt{q/p}\;B\,\right|^2=(1+p/q)\,|\,A\,|^2+(1+q/p)\,|\,B\,|^2. \qquad (3.44)$$

This equation is equivalent to (3.40) since

$$1/p+1/q=1\implies[1+p/q=p \quad \& \quad 1+q/p=q].$$

According to [21] , the equivalence between (3.40) and (3.41) follows if the former inequality is divided by $p\,q$ and $1/q=_{\text{not}}\alpha$. The coefficients in the right side of (3.41) , after dividing it by $p\,q$, become $(\,p+q)\,/\,(\,p\,q)\,q=1/\,q$ and (respectively) $(\,p+q)\,/\,(\,p\,q)\,p=1\,/\,p=1-1\,/\,q$. These two equalities follow from our previous equalities of (3.33) . Hence, the right side of (3.41) actually becomes

$= \alpha\ |A|^2 + (1-\alpha\ )\ |B|^2$. The first term in the left side of (3.41) becomes $\alpha\ (1-\alpha\ )\ |A-B|^2$. As regards the second term in the left side of (3.41), its multiplication by $1/(pq) = (1/\sqrt{pq}\ )^2$ turns the coefficients inside $|\ldots|$ into

$$\sqrt{p/q}\,/\sqrt{pq} = 1/q = \alpha\ \&\ \sqrt{q/p}\,/\sqrt{pq} = 1/p = 1-\alpha\ . \tag{3.45}$$

Hence, the identity (3.41) is equivalent to (3.40) .

Remark 3.9. The author F. Zhang asserts that the identity (3.41) gives immediately the inequality

$$|\alpha A + (1-\alpha)B|^2 \le \alpha\,|A|^2 + (1-\alpha)\,|B|^2, \tag{3.46}$$

that is, the *square-convexity inequality.* This inequality would follow by the positivity of $\alpha\ (1-\alpha)\ |A-B|^2$, since $|A-B|^2$ is positive and $\alpha\ (1-\alpha\ ) \in [0\ ,\ 1]$. Hence, if this term is omitted (removed) from the left side of (3.48) , the remaining term becomes less than the right side. For the equality case in (3.29) , the second term in the left side of (3.40) should vanish, that is

$$\sqrt{p/q}\ A + \sqrt{q/p}\ B\ = 0 \Longleftrightarrow (p/q)\ A + B = 0 \Longleftrightarrow B = (1-p)\ A\ ,$$

which is just the equality case in O. Hirzallah's *Corollary 1* of [10] .

As regards the proof that inequality (3.29) follows from the identity (3.40) , F. Zhang notices that it should be proved that

$$|(1-p)A - B|^2 \le \left|\sqrt{p/q}\ A + \sqrt{q/p}\ B\ \right|^2$$

when $1 \le p \le 2$. Before continuing, let us see an equivalent form of this inequality ; it follows by the replacements $\sqrt{p/q} \to \sqrt{p-1}$ and $\sqrt{q/p} \to \sqrt{q-1}$:

$$|(1-p)A - B|^{\,2} \le \left|\sqrt{p-1}\ A + \sqrt{q-1}\ B\ \right|^2 . \tag{3.47}$$

But the author states and proves two Lemmas that lead to a more general inequality :

Lemma 1 ([18]) *Let* $A,\ B \in \mathcal{B}(\boldsymbol{H}\ )$. *If* $a,\ b > 0$, $c \in \mathbb{R}$ *and* $a\,b \ge c^2$, *then*

$$a|A|^2 + b|B|^2 + c(A^*B + B^*A) \ge 0\ . \tag{3.48}$$

The proof of (3.48) makes use of 2-by-2 matrices with both scalar and operator entries. The latter operator matrices were considered in P. Halmos's monograph

[6] as well in other sources. A 2-by-2 operator matrix acts on the direct sum of a Hilbert space by itself,

$$\boldsymbol{H} \oplus \boldsymbol{H} = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ y \end{bmatrix} : x \,,\; y \in \boldsymbol{H} \right\} \tag{3.49}$$

Such a direct sum is a Hilbert space as well. An operator acting on it, involving four H-space operators A, B, C, D looks like and it is defined by

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}, \quad \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} A\,x + B\,y \\ C\,x + Dy \end{bmatrix}.$$

The 2-by-2 matrix with real entries in the statement of this Lemma is positive, that is

$$\begin{bmatrix} a & c \\ c & b \end{bmatrix} \geq 0 \quad \text{since} \quad a\,b \geq c^2. \tag{3.50}$$

Another positive matrix (with operator entries) is

$$\begin{bmatrix} A^* \\ B^* \end{bmatrix} \begin{bmatrix} A & B \end{bmatrix} = \begin{bmatrix} |\,A\,|^2 & A^*B \\ B^*A & |\,B\,|^2 \end{bmatrix} \geq 0 \Longrightarrow \begin{bmatrix} a\,|\,A\,|^2 & c\,A^*B \\ c\,B^*A & b\,|\,B\,|^2 \end{bmatrix} \geq 0\,. \tag{3.51}$$

Thus,

$$\begin{bmatrix} I & I \end{bmatrix} \begin{bmatrix} a\,|\,A\,|^2 & c\,A^*B \\ c\,B^*A & b\,|\,B\,|^2 \end{bmatrix} \begin{bmatrix} I \\ I \end{bmatrix} = a\,|\,A\,|^2 + b\,|\,B\,|^2 + c\,(\,A^*B + B^*A\,) \geq 0\,. \tag{3.52}$$

Before recalling another lemma of [21], let us notice that the positivity of the 2-by-2 matrix (over $\mathbb{R}$) means that it is positive definite. The notations in the subsequent equations slightly differ from those of [21] . For instance, F. Zhang's notation $(\,I\,,\,I\,)$ looks like the one specific to an ordered pair, but it denotes (in fact) the row vector $[\,I \quad I\,]$, with $I =$ the identity operator (also denoted as **1**). As regards the first operation which occurs in (3.52), it is not an actual (matrix) product but a conventional way to obtain a 2-by-2 matrix from the "product" of a 2-by-1 matrix (column vector) with a 1-by-2 matrix (row vector). We used a similar way for defining the matrix of a BLF − a bilinear form f in a basis $A = [\,a_1 \;\; \ldots \;\; a_i \;\; \ldots \;\; a_j \;\; \ldots \;\; a_n\,]$ spanning an $n-$ dimensional vector space V (in our textbook [3]) : in such a basis, written as a row of vectors, the matrix is defined as $F_A = f\,(A^{\mathrm{T}},\, A) = [\; f\,(\,a_i\,,\, a_j\,)]_{n \times n}$. The next **Lemma 2** of [18] states that :

If $A,\, B \in \mathcal{B}(\boldsymbol{H})$ *and* $x,\, y,\, s,\, t \in \mathbb{R}$ *such that* $|\,x\,| \leq |\,s\,|,\, |\,y\,| \leq |\,t\,|$ & $x\,t = s\,y$ *then*

$$|\,x\,A + y\,B\,|^2 \leq |\,s\,A + t\,B\,|^2. \tag{3.53}$$

This inequality involving two operators and four real parameters follows by expanding the two sides of (3.53), moving the terms from the left to the right side and then applying *Lemma 1* with

$$a = s^2 - x^2, \quad b = t^2 - y^2 \quad \text{and} \quad c = s\,t - x\,y\,. \tag{3.54}$$

We have earlier seen that $1 \le p \le q \implies 1 \le p \le 2$. Taking $x = p - 1$, $y = 1$, $s = \sqrt{p\,/\,q}$, $t = \sqrt{q\,/\,p}$ in (3.53), the implication $|\,(1-p)\,A - B\,|^2 \le \left|\sqrt{p\,/\,q}\;A + \sqrt{q\,/\,p}\;B\,\right|^2 \implies$ (3.22) immediately follows.

The author also remarks that his *Theorem 2*, that is the identity (3.40), can generate a whole variety of inequalities similar to O. Hirzallah's *Theorem 1*, that is (3.22). Indeed, putting $s = \sqrt{p/q}$, $t = \sqrt{q/p}$, for any real numbers x & y satisfying $x\,q = y\,p$ and $x^2 \le p\,/\,q = p - 1$ in (3.53), it follows that

$$|\,A - B\,|^{\,2} + |\,x\,A + y\,B\,|^{\,2} \le p\,|\,A\,|^{\,2} + q\,|\,B\,|^{\,2}. \tag{3.55}$$

Remark 3.10. (i) The proof of Lemma 2, that is of inequality (3.53), follows from the expansion of the two sides in (3.53) and from

$$|\,x\,| \le |\,s\,|\,,\quad |\,y\,| \le |\,t\,| \quad \& \quad x\,t = s\,y \implies x^2 \le s^2\,, \quad y^2 \le t^2 \implies$$

$$\implies \begin{cases} a\,b = \;\ldots\; = s^2\,t^2 + x^2\,t^2 - (\,s^2\,y^2 + t^2\,x^2), \\ \qquad\quad c^2 = s^2\,t^2 + x^2\,t^2 - 2\,s\,t\,x\,y \end{cases} \implies \ldots \implies a\,b \ge c^2. \tag{3.56}$$

It would follow that inequality (3.53) holds. However, the implications in the above formulas (3.56) are not quite immediate. We are going to give more details in our Proposition which follows.

(ii) Another extension of the Bohr inequality, which is implied by (3.55), holds for $1 \le p \le q$, $x = (\,p-1\,)^k$ and $y = (\,p-1\,)^{k-1}$, where k is any positive integer :

$$|\,A - B\,|^2 + |\,(p-1)^k\;A + (p-1)^{k-1}\;B\,|^2 \le p\,|\,A\,|^2 + q\,|\,B\,|^2. \tag{3.57}$$

The implication from (3.55) to (3.57) is true since

$$x = (\,p-1)^k = y\;(\,p-1) = (\,p\,/\,q\,)\,y \implies x\,q = y\,p\,, \tag{3.58}$$

and, with the notations in *Lemma 2* and *Theorem 2* of [18] , also taking into account that $1 \le p \le q \implies 1 \le p \le 2 \implies p - 1 \le 1$,

$$\begin{cases} s = \sqrt{p\,/\,q} \implies s^2 = p\,/\,q\,, \\ x = (\,p-1)^k \implies x^2 = (\,p-1)^{2\,k} \end{cases} \implies$$

$$\Longrightarrow\ x^2 = (p-1)^{2k-1}(p-1) \le p-1 = p/q = s^2.$$

Similarly, $y^2 = (p-1)^{2k-2}$ and $t^2 = q/p \Longrightarrow$

$$\Longrightarrow y^2 = (p-1)^{2k}/(p-1)^2 = (p-1)^{2k}\,(q/p) \le t^2.$$

Therefore, the hypotheses in *Lemma 2* are satisfied, inequalities (3.55) and (3.56) hold, and they imply (3.57) . It is obvious that inequality (3.57) reduces to (3.28) when $k = 1$.

Proposition 3.4. *(Explicit proof of inequality* $ab \ge c^2$ *in* (3.56)*).*

Proof. It follows from the statement of F. Zhang's *Lemma 2*, namely from

$$|x| \le |s|,\quad |y| \le |t| \quad \& \quad xt = sy, \tag{3.59}$$

that $|x\,t| = |s\,y| \Longrightarrow |x|\,|t| = |s|\,|y|$. But the two inequalities of (3.59) imply

$$|x|\,|y| \le |x|\,|t| \le |s|\,|t| \Longrightarrow |x|^2\,|y|^2 \le |s|^2\,|t|^2 \Longrightarrow x^2\,y^2 \le s^2\,t^2. \tag{3.60}$$

Next, the expressions of a & b in (3.54) lead to

$$a\,b = (s^2 - x^2)\,(t^2 - y^2) = s^2\,t^2 + x^2\,y^2 - (x^2\,t^2 + s^2\,y^2). \tag{3.61}$$

Expression (3.61) plus the equation in (3.59) lead to

$$a\,b = s^2\,t^2 + x^2\,y^2 - 2\,t^2\,x^2 = s^2\,t^2 + x^2\,y^2 - 2\,s^2\,y^2. \tag{3.62}$$

On another hand, $c = st - x\,y \Longrightarrow c^2 = (st - x\,y)^2 = s^2t^2 + x^2\,y^2 - 2\,st\,x\,y =$

$$= s^2\,t^2 + x^2\,y^2 - 2\,(x\,t)(s\,y) = s^2\,t^2 + x^2\,y^2 - 2\,s^2t^2. \tag{3.63}$$

Now, if the equation (3.63) is side-by-side subtracted from (3.61), that is leftmost / rightmost sides, it follows that

$$a\,b - c^2 = \ \dots\ = -\,2\,s^2\,y^2 + 2\,s^2\,t^2 = 2\,s^2(t^2 - y^2) \underset{(3.59)}{\ge} 0$$

and this proves that inequality $a\,b \ge c^2$ holds. □

4 Concluding remarks

This paper offers a survey of some inequalities defined on Hilbert spaces. The literature dedicated to this area of research is quite rich. We found more than 40 papers and monographs approaching this subject, but it was not possible to make reference to more than 21 of them. Several classical inequalities like those due to Harald Bohr (of 1924), F. Rellich (1950),T. Kato (1952) attracted the interest of many authors who established equivalent expressions and (especially) extensions/generalizations of such inequalities. Some authors deserve mention in this respect: F. Hansen, F. Kittaneh, O. Hirzallah, J. Pečarić, F. Zhang are only a few of them. In **Section 2** we recalled a series of basic definitions and notations on Hilbert space operators, classification of some H-space operators, etc. The subsequent **Section 3** with its subsections present some ways to obtain operator inequalities by specific methods/ways. We approached several inequalities obtained from inequalities on the algebra of operators and on the field of (complex) scalars $\mathbb{C}$. In subsection **3.1** we dealt with operator inequalities induced by operator positivity and we checked that $A \geq B$ is an actual partial order relation. We showed, in subsection **3.2**, that T.Kato's relation $S \ll T$ is a partial order as well. Inequalities of Cauchy-Schwarz type are presented in subsection **3.3**. The next two subsections **3.4**, **3.5** were dedicated to Harald Bohr's inequality and its many equivalent expressions and generalizations. In **3.5** we recalled F. Zhang's method to obtain Bohr-type inequalities from certain identities. In this *Section 3* we gave some completions to a series of proofs, we tried to find connections between certain inequalities, in ten Remarks and four Propositions. It may be considered, as our *main results*, the details added to O. Hirzallah's proofs of his Theorem 1 and Corollary 1 of [10], mainly in the three parts of our Proposition 3.3. Other completions were given in *Remark 3.10*, on F. Zhang's formula (3.53) and its connections with earlier considered Bohr-type inequalities, we also proved the inequality $ab > c^2$ in the last Proposition 3.4.

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