

Equivalent definitions for connections in higher order tangent bundles

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Abstract: We establish a one to one correspondence between the connections $C^{(k-1)}$ (in the bundle $T^k M \rightarrow M$, used by R. Miron in his work on higher order spaces) and $C^{(0)}$ (in the affine bundle $T^k M \rightarrow T^{k-1} M$, used for example in [3]).

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The authors dedicate the paper to Academician Radu Miron on the occasion of his 90th birthday

1 Introduction

We establish a correspondence one to one between the connections $C^{(k-1)}$ (in the bundle $T^k M \rightarrow M$, used by R. Miron) and $C^{(0)}$ (in the affine bundle $T^k M \rightarrow T^{k-1} M$, used for example in [3]). The coefficients of $C^{(0)}$ are called, following Miron, as dual coefficients, but not related as the coefficients of a connection. In this case we use notations more adequate to our setting, different from [1] or [4]. A synthetic review of some problems involving higher order connections is performed in [2].

2 Some notions regarding higher order spaces

The higher order space $T^k M$ of a manifold M can be viewed as a fibered manifold in k ways, over the bases $T^i M$, $0 \leq i \leq k-1$. Firstly, let us consider the extreme cases $i = 0$ and $i = k-1$, i.e. the fibered manifolds $T^k M \xrightarrow{\pi_0} M$ and $T^k M \xrightarrow{\pi_{k-1}} T^{k-1} M$.

The fibered manifold $T^k M \xrightarrow{\pi_0} M$ was systematically used by R. Miron in the study of non-linear connections, semi-sprays and Lagrangians of order k (for example in [4]).

The affine bundle $T^k M \xrightarrow{\pi_{k-1}} T^{k-1} M$ was used in the papers [3] and [8], in the study of affine Hamiltonians of order k , a natural dual counterpart of Lagrangians of order k .

The higher order space $T^k M$ is defined according by the change rule of the local coordinates given by $ky^{(k)i'} = k \frac{\partial x^{i'}}{\partial x^i} y^{(k)i} + \Gamma_U^{(k-1)}(y^{(k-1)i'})$, where $\Gamma_U^{(k-1)} = y^{(1)i} \frac{\partial}{\partial x^i} + 2y^{(2)i} \frac{\partial}{\partial y^{(1)i}} + \dots + (k-1)y^{(k-1)i} \frac{\partial}{\partial y^{(k-2)i}}$ is only a local vector field. The Liouville vector field has the local form $\overset{k}{\Gamma} = y^{(1)i} \frac{\partial}{\partial y^{(1)i}} + \dots + y^{(k)i} \frac{\partial}{\partial y^{(k)i}}$ and it is a global vector field.

Proposition 2.1. *The fibered manifold $(T^k M, \pi_k, T^{k-1} M)$ is an affine bundle, for $k \geq 2$.*

Notice that the coordinates $y^{(p)i}$ are in accordance with [4]-[7].

In the sequel we use the dual k -structures $J : TT^k M \rightarrow TT^k M$ and $J^* : T^*T^k M \rightarrow T^*T^k M$, given by:

$$J = \frac{\partial}{\partial y^{(1)i}} \otimes dx^i + \frac{\partial}{\partial y^{(2)i}} \otimes dy^{(1)i} + \dots + \frac{\partial}{\partial y^{(k)i}} \otimes dy^{(k-1)i}, \quad (\text{J})$$

$$J^* = dy^{(k-1)i} \otimes \frac{\partial}{\partial y^{(k)i}} + dy^{(k-2)i} \otimes \frac{\partial}{\partial y^{(k-1)i}} + \dots + dx^i \otimes \frac{\partial}{\partial y^{(1)i}}, \quad (\text{J}^*)$$

or

$$\begin{aligned} \frac{\partial}{\partial x^i} &\xrightarrow{J} \frac{\partial}{\partial y^{(1)i}} \xrightarrow{J} \dots \xrightarrow{J} \frac{\partial}{\partial y^{(k)i}} \xrightarrow{J} 0, \\ dy^{(k)i} &\xrightarrow{J^*} dy^{(k-1)i} \xrightarrow{J^*} \dots \xrightarrow{J^*} dx^i \xrightarrow{J^*} 0. \end{aligned}$$

It is easy to see that J and J^* are (globally defined) endomorphisms on $T^k M$, $\text{rank } J = \text{rank } J^* = km$.

The vector bundle canonically associated with the affine bundle $(T^k M, \pi_k, T^{k-1} M)$ is the vector bundle $q_{k-1}^* TM$, where $q_{k-1} : T^{k-1} M \rightarrow M$ is $q_{k-1} = \pi_1 \circ \pi_2 \circ \dots \circ \pi_{k-1}$. The fibered manifold $(T^k M, q_k, M)$ is used in [4], [7] in the study of the geometrical objects of order k on M , in particular in the study of Lagrangians of order k on M .

The tensors defined on the fibers of the vertical vector bundle $V_{k-1}^k M \rightarrow T^k M$ of the affine bundle $(T^k M, \pi_k, T^{k-1} M)$, or on the fibers of an open fibered submanifold of $V_{k-1}^k M \rightarrow T^k M$, are called d -tensors of order k on M . We denote by $V_j^k M \rightarrow T^k M$ the vertical bundle of the bundle $T^k M \rightarrow T^j M$, where $T^{(0)} M = M$. Obviously $TT^k M \supset V_0^k M \supset V_1^k M \supset \dots \supset V_{k-1}^k M$. Their polar bundles are $W_j^k M = (V_j^k M)^\circ$, thus $W_0^k M \subset W_1^k M \subset \dots \subset W_{k-1}^k M \subset T^*T^k M$.

It is easy to see that $\text{Im } J = V_0^k$, $\ker J = V_{k-1}^k$, $\text{Im } J^* = W_{k-1}^k$, $\ker J^* = W_0^k$.

Notice that the local coordinates $(x^i, y^{(1)i}, \dots, y^{(k)i})$ on $T^k M$ change according to the rules:

$$\begin{aligned} y^{(1)i'} &= y^{(1)i} \frac{\partial x^{i'}}{\partial x^i}, \\ 2y^{(2)i'} &= y^{(1)i} \frac{\partial y^{(1)i'}}{\partial x^i} + 2y^{(2)i} \frac{\partial y^{(1)i'}}{\partial y^{(1)i}}, \\ &\vdots \\ ky^{(k)i} &= y^{(1)i} \frac{\partial y^{(k-1)i'}}{\partial x^i} + \dots + ky^{(k)i} \frac{\partial y^{(k-1)i'}}{\partial y^{(k-1)i}}. \end{aligned} \tag{2.1}$$

Local bases of modules of sections in $\Gamma(V_j^k M)$ and $\Gamma(W_j^k M)$ are $\left\{ \frac{\partial}{\partial y^{(j)i}}, \dots, \frac{\partial}{\partial y^{(k-1)i}} \right\}$ and $\{dx^i, dy^{(j)i}, \dots, dy^{(j-1)i}\}$ respectively.

3 Connections

A non-linear connection in the affine bundle $(T^k M, \pi_k, T^{k-1} M)$ is a left splitting $C^{(k-1)}$ of the inclusion $V_{k-1}^k M \xrightarrow{I_{k-1}} TT^k M$, i.e. a vector bundle map $TT^k M \xrightarrow{C^{(k-1)}} V_{k-1}^k M$ such that $C^{(k-1)} \circ I_{k-1} = 1_{V_{k-1}^k M}$. Using local coordinates, $C^{(k-1)}$ has the local form:

$$\begin{aligned} (x^i, y^{(1)i}, \dots, y^{(k)i}, Y^{(0)i}, Y^{(1)i}, \dots, Y^{(k)i}) &\xrightarrow{C^{(k-1)}} \\ (x^i, y^{(1)i}, \dots, y^{(k)i}, Y^{(k)i} + M_{(0)j}^i Y^{(0)j} + \dots + M_{(k-1)j}^i Y^{(k-1)j}). \end{aligned} \tag{3.1}$$

A non-linear connection in the bundle $(T^k M, \pi_k, M)$ is a left splitting $C^{(0)}$ of the inclusion $V_0^k M \xrightarrow{I_0} TT^k M$, i.e. a vector bundle map $TT^k M \xrightarrow{C^{(0)}} V_0^k M$ such that $C^{(0)} \circ I_0 = 1_{V_0^k M}$. Using local coordinates, $C^{(0)}$ has the local form:

$$\begin{aligned} (x^i, y^{(1)i}, \dots, y^{(k)i}, Y^{(0)i}, Y^{(1)i}, \dots, Y^{(k)i}) &\xrightarrow{C^{(0)}} \\ (x^i, y^{(1)i}, \dots, y^{(k)i}, Y^{(1)i} + N_j^{(1)i} Y^{(0)j}, \dots, Y^{(k)i} + N_j^{(k)i} Y^{(0)j}). \end{aligned} \tag{3.2}$$

We are going to establish a one to one correspondence between connections $C^{(0)}$ and $C^{(k-1)}$.

Using constructions performed in [4] (see also [5]), we can perform the following constructions using $C^{(0)}$.

If one denotes by $N_0 \subset TT^k M$ the horizontal distribution of $C^{(0)}$, then the successive distributions:

$$N_1 = J(N_0), \dots, N_{k-1} = J^{k-1}(N_0), V_{k-1}^k M = J^k(N_0) \quad (3.3)$$

give the decomposition:

$$TT^k M = N_0 \oplus N_1 \oplus \dots \oplus N_{k-1} \oplus V_{k-1}^k M \quad (3.4)$$

and the corresponding projectors denoted by h, v_1, \dots, v_k . A local base $\mathcal{B}_0 = \left\{ \frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N_i^{(1)j} \frac{\partial}{\partial y^{(1)j}} - \dots - N_i^{(k-1)j} \frac{\partial}{\partial y^{(k-1)j}} \right\}$ for $\Gamma(N_0)$ generate the bases $\mathcal{B}_1 = J(\mathcal{B}_0) \subset \Gamma(N_1), \dots, \mathcal{B}_{k-1} = J^{k-1}(\mathcal{B}_0) \subset \Gamma(N_{k-1}), \mathcal{B}_k = J^k(\mathcal{B}_0) \subset \Gamma(V_{k-1}^k M)$ and $\mathcal{B} = \mathcal{B}_0 \cup \mathcal{B}_1 \cup \dots \cup \mathcal{B}_k$ is a local base for $\mathcal{X}(T^k M)$, called a *Berwald base* (see [1]).

Let us consider a dual base of \mathcal{B} , $\mathcal{B}^* = \{\delta x^i = dx^i, \delta y^{(1)i}, \dots, \delta y^{(k)i}\}$, where

$$\delta y^{(k)i} = M_{(k)j}^i dx^j + M_{(k-1)j}^i dy^{(1)j} + \dots + M_{(1)j}^i dy^{(k-1)j} + dy^{(k)i}$$

and $\delta y^{(k-1)i} = J^* \delta y^{(k)i}, \dots, \delta y^{(1)i} = (J^*)^{k-1} \delta y^{(k)i}, \delta x^i = dx^i = (J^*)^k \delta y^{(k)i}$. The *dual coefficients* of the connection $C^{(0)}$ are the above coefficients of $\delta y^{(k)i}$. The duality conditions give a link between the coefficients $N_j^{(\alpha)i}$ and $M_{(\beta)j}^i$, given by:

$$\begin{aligned} N_j^{(1)i} &= M_{(1)j}^i, \\ N_j^{(2)i} &= M_{(2)j}^i - M_{(1)p}^i N_j^{(1)p}, \\ &\vdots \\ N_j^{(k)i} &= M_{(k)j}^i - M_{(k-1)p}^i N_j^{(1)p} - \dots - M_{(1)p}^i N_j^{(k-1)p}, \end{aligned} \quad (3.5)$$

and conversely:

$$\begin{aligned} M_{(1)j}^i &= N_j^{(1)i}, \\ M_{(2)j}^i &= N_j^{(2)i} + M_{(1)p}^i N_j^{(1)p} \\ &\vdots \\ M_{(k)j}^i &= N_j^{(k)i} + M_{(k-1)p}^i N_j^{(1)p} + \dots + M_{(1)p}^i N_j^{(k-1)p}. \end{aligned} \quad (3.6)$$

The coefficients M change according to the rule:

$$\begin{cases} M_{(1)i}^j \frac{\partial x^{i'}}{\partial x^j} = M_{(1)j'}^{i'} \frac{\partial x^{j'}}{\partial x^i} + \frac{\partial y^{(1)i'}}{\partial x^i}, \\ \vdots \\ M_{(k)i}^j \frac{\partial x^{i'}}{\partial x^j} = M_{(k)j'}^{i'} \frac{\partial x^{j'}}{\partial x^i} + M_{(k-1)j'}^{i'} \frac{\partial y^{(1)j'}}{\partial x^i} + \dots + M_{(1)j'}^{i'} \frac{\partial y^{(k-1)j'}}{\partial x^i} + \frac{\partial y^{(k)i'}}{\partial x^i}. \end{cases} \quad (3.7)$$

The change rule of $\delta y^{(k)i}$ is $\delta y^{(k)i'} = \frac{\partial x^{i'}}{\partial x^i} \delta y^{(k)i}$. By a straightforward computation one can prove that a connection $C^{(k-1)}$ that has the local form (3.1) can be defined. Its horizontal projector has the form:

$$\begin{aligned} & h \left(Y^{(0)i} \frac{\partial}{\partial x^i} + Y^{(1)i} \frac{\partial}{\partial y^{(1)i}} + \dots + Y^{(k-1)i} \frac{\partial}{\partial y^{(k-1)i}} \right) \\ &= Y^{(0)i} \left(\frac{\partial}{\partial x^i} - M_{(k)i}^j \frac{\partial}{\partial y^{(k)i}} \right) + Y^{(1)i} \left(\frac{\partial}{\partial y^{(k-1)i}} - M_{(1)i}^j \frac{\partial}{\partial y^{(k)i}} \right) + \dots \\ &+ Y^{(k-1)i} \left(\frac{\partial}{\partial y^{(k-1)i}} - M_{(1)i}^j \frac{\partial}{\partial y^{(k)i}} \right). \end{aligned}$$

It can be also proved that conversely, given a connection $C^{(k-1)}$ having the coefficients $(M_{(\alpha)j}^i)$, then a (non-linear) connection $C^{(0)}$ with coefficients $(N_j^{(\alpha)i})$ given by relations (3.5) is obtained, having $(M_{(\alpha)j}^i)$ as dual coefficients.

A link between the connections $C^{(0)}$ and $C^{(k-1)}$ can be obtained without involving coordinates, as follows. Let us consider a given connection $C^{(0)}$, thus the distribution N_0 is given. We can construct the distributions (3.3) and the distribution (3.4). We have $TT^k M = N_k \oplus V_{k-1}^k M$, where $N_k = N_0 \oplus N_1 \oplus \dots \oplus N_{k-1}$. The polar subbundle $N_k^\circ \subset T^*T^k M$ (given by forms in $T^*T^k M$ that are null on N_k) is canonically isomorphic with the dual of the vertical bundle $V_{k-1}^k M$, thus the inclusion $N_k^\circ \subset T^*T^k M$ reverses by duality to a vector bundle epimorphism $TT^k M \xrightarrow{C^{(k-1)}} V_{k-1}^k M$ that defines the connection $C^{(k-1)}$.

In order to obtain $C^{(0)}$ from $C^{(k-1)}$ we proceed analogously, following a converse way: The vector bundle epimorphism $TT^k M \xrightarrow{C^{(k-1)}} V_{k-1}^k M$ that defines the connection $C^{(k-1)}$ reverses by duality to an inclusion $W^k = (V_{k-1}^k M)^* \subset T^*T^k M$. We consider the subbundles of $T^*T^k M$: $W^{k-1} = J^*(W^k), \dots, W^1 = (J^*)^{k-1}(W^k), W^0 = (J^*)^k(W^k)$. We have that $T^*T^k M = W^0 \oplus \dots \oplus W^k$ and $W^0 = (V_0^k M)^\circ$. Then considering $\mathcal{U} = W^1 \oplus \dots \oplus W^k$ we have $T^*T^k M = \mathcal{U} \oplus (V_0^k M)^\circ$ and \mathcal{U} is canonically isomorphic with $(V_0^k M)^*$. Thus the inclusion $\mathcal{U} \subset T^*T^k M$ reverses by duality to a vector bundle epimorphism $TT^k M \xrightarrow{C^{(0)}} V_0^k M$ that defines the connection $C^{(0)}$.

All these prove the following interpretation of the dual coefficients of a non-linear connection of order k , which is the main result of the paper.

Theorem 3.1. *There is a one to one correspondence between the connections $C^{(0)}$ in the bundle $T^k M \xrightarrow{\pi_Q} M$ and connections $C^{(k-1)}$ in the bundle $T^k M \xrightarrow{\pi_{k-1}^{-1}} T^{k-1} M$.*

As we have seen, the correspondence given by the above Proposition associates, in local coordinates, but also in an invariant form, free of coordinates, a connection

$C^{(k-1)}$ with a connection $C^{(0)}$ that has the dual coefficients exactly the coefficients of $C^{(k-1)}$.

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