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Semirings with an Almost Division Algorithm

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Abstract. The notion and some algebraic properties of additively absorptive subsemirings of order $t \geq 1$ (t a fixed integer) in a Euclidean semiring as a natural extention of additively absorptive subrings of Euclidean rings is discussed. For a fixed integer $t \geq 1$, a proper subsemiring D of a Euclidean semiring R with Euclidean function ϕ, is said to be additively absorptive (or simply, absorptive) of order t in R; if for each f in R, there exists h in D and g in R such that $f = h + g$ and $1 \leq \phi(g) \leq t$. The main result of the paper states that if I is a k-ideal of an absorptive subsemiring D of order t in R as a member of a subclass of Euclidean semirings, then I can be generated by $(t+1)$ or fewer elements. In addition, if I contains an element of ϕ value equals to $i+\phi(I)$ for some $1 \leq i \leq t$ with $\phi(I) = inf\{\phi(f) \mid f \in I\}$, then I cannot be a principal ideal in D. In this case, D can not be a Bezout semiring (i.e., a semiring in which every finitely generated ideal is principal).

Keywords: Euclidean semiring, subtractive ideal (= k-ideal), additively absorptive subsemiring of order t.

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1 Introduction

In this paper (unless otherwise indicated), R always denotes a commutative semiring with identity $1 \neq 0$, and $0a = 0$ for all a in R. A subsemiring of a semiring R is a semiring that contains 1_R and 0_R. A nonempty subset I of a semiring R will be called an ideal if $a, b \in I$ and r in R implies $a + b$ in I and ra in I. A subtractive ideal (= k-ideal) I is an ideal such that if $x, x + y \in I$, then y in I (so (0) is a k-ideal of R). The k-closure $cl(I)$ of I is defined by $cl(I) = \{a \in R \mid a + c = d \text{for some } c, d \ \in I\}$ is an ideal of R satisfying $I \subseteq cl(I)$ and $cl(cl(I)) = cl(I)$. So an ideal I of R is a k-ideal if and only if $I = cl(I)$. note that in [2], Golan uses the term "subtractive ideal", but in the literature of semirings, authors use equivalently the term "k-ideal" as well. In this work, we use preferably both terminologies as well without mentioning that they are the same. For the sake of completeness, we state some definitions and notions used throughout to keep this paper as self-contained as possible. Also, for a detailed study of semirings and plenty of examples (counterexamples) together with many related references, reader is referred to [2]. Here, before stating the definition and main results, we write and reflect the definition of a (left) Euclidean semiring together wit some examples (counterexample) exactly as given in Chapter 11 of [2] as follows.

Let N be the set of all nonnegative integers. A left Euclidean norm δ defined on a semiring R is a function $\delta : R\backslash\{0\} \to N$ satisfying the following condition:

$(*)$ If a and b are elements of R with $b \neq 0$, then there exist elements q and r of R satisfying $a = qb + r$ with $r = 0$ or $\delta(r) < \delta(b)$.

A right Euclidean norm is defined similarly, except that in Condition $(*)$ we have $a = bq + r$. A semiring R is left [resp. right] Euclidean if and only if there exists a left [resp. right] Euclidean norm defined on R. For commutative semirings, needless to say, the notions of left and right Euclidean norm coincide.

Remark 1. *In some algebra books such as [3], the definition of a Euclidean ring contains another extra condition which is called submultiplicative condition and states $\phi(ab) \geq \phi(a)$ for all elements a and b in R with $ab \neq 0$. In Proposition 11.10 of [2], it is shown that this condition is not necessary to be stated since it can be proved for every Euclidean semiring there exists a Euclidean function (norm) that satisfys the submultiplicative condition. Thus, we always assume that a Euclidean semiring satisfies the submultiplicative condition.*

Example 1. *Since every ring is a semiring, thus every Euclidean ring is an example of a Euclidean semiring.*

Example 2. *The semiring N of nonnegative integers is Euclidean if we define the Euclidean norm δ by $\delta : n \mapsto n$ or $\delta : n \mapsto n^2$.*

Example 3. *Let $S[t]$ be the semiring of polynomials in the indeterminate t over a division semiring S and let $\equiv$ be the congruence relation on $S[t]$ defined by sum $a_i t^i \equiv \sum b_i t^i$ if and only if $a_1 t + a_0 = b_1 t + b_0$. Let R be the factor semiring $S[t]/\equiv$. Then there exists a left Euclidean norm $\delta : R\backslash\{0\} \to N$ defined by setting $\delta(\sum a_i t^i/\equiv) = 1$ if $a_1 \neq 0$ and equals to 0 if $a_1 = 0$ but $a_0 \neq 0$.*

Example 4. *Let R be the subsemiring of Q^+ (the set of nonnegative rationals) defined by $R = \{q \in Q^+ \mid q = 0\text{or } q \geq 1\}$ and suppose that we have a left Euclidean norm $\delta : R\backslash\{0\} \to N$. Let $0 < a < b$ be elements of R. If $\delta(a) \geq \delta(b)$, then there would have to exist elements q and r of R satisfying $a = qb + r$, where $r = 0$ or $\delta(r) < \delta(b)$. But $a < b$ implies that $a < qb$ for all $0 \neq q \in R$ and $a = 0b + r$ leads to the contradiction $\delta(a) = \delta(r) < \delta(b)$. Thus, $a < b$ implies that $\delta(a) < \delta(b)$ for all $0 \neq a, b \in R$. Hence, $R\backslash\{0\}$ is order-isomorphic to the subset $im(\delta)$ of N, which is impossible. Thus, no left Euclidean norm can be defined on R, and so R is not a left Euclidean semiring.*

Example 5. *The set $2N$ of all nonnegative even integers is a subtractive ideal of the semiring of all nonnegative integers. It is not strong since $3 + 5$ in $2N$ while neither 3 nor 5 belong to $2N$. An ideal I of a semiring R is said to be strong whenever $a + b \in I$ implies a and b are in I for all a and b in R. For a complete study of this, see Example 5.4 in [2].*

In this work, we extend the work of Rahimi [5] as a natural extention of Euclidean rings to Euclidean semirings. In Section 2, we define and study the concept and some algebraic properties of additively absorptive subsemirings of order $t \geq 1$ (t a fixed integer) in a Euclidean semiring as a natural extention of additively absorptive subrings in Euclidean rings. Also, examples of both absorptive and nonabsorptive subrings of $k[x]$ (the ring of polynomials over a field k) are given. In Section 3, the main results of the paper are as follows. If D is an absorptive and subtractive subsemiring of order t in a Euclidean semiring R with a Euclidean function ϕ which satisfies the following conditions:

(1) for all $f, g, r \in R$, $\phi(fg) = \phi(f) + \phi(g)$; and

(2) $\phi(r) \leq \phi(g)$ implies $\phi(r + g) = \phi(g)$,

then for each $f, g \in D$, $g \neq 0$, there exist $q, r \in D$ such that $f = qg + r$ with $r = 0$ or $\phi(r) < \phi(g)$ or $\phi(r) = i + \phi(g)$ for some $1 \leq i \leq t$. Furthermore, it is shown that if $I \neq 0$ is a k-ideal of D (not necessarily subtractive), then I can be generated by $(t + 1)$ or fewer elements. In addition, if I contains an element of ϕ value equals to $i+\phi(I)$ for some $1 \leq i \leq t$ with $\phi(I) = inf\{\phi(f) \mid f \in I\}$, then I cannot be a principal ideal in D. Finally, in the last section, we explore this subject through a general overview and show that the semiring D of Part (ii) in Theorem 7 is not Bezout which consequently can never be a Hermite semiring. . .

2 Absorptive Subsemirings

In this section we investigate the concept and some algebraic properties of additively absorptive subsemirings of order $t \geq 1$ (t a fixed integer) in a Euclidean semiring as a natural extention of additively absorptive subrings in Euclidean rings.

Definition 1. *Let R be a Euclidean semiring with Euclidean function ϕ, and assume $t \geq 1$ is a fixed integer. A proper subsemiring D of R is said to be additively absorptive (or simply, absorptive) of order t in R, if for each f in R there exists h in D and g in R such that $f = h + g$ and $1 \leq \phi(g) \leq t$.*

Remark 2. *Note that in [5], Rahimi has defined the absorptive subring of a commutative Euclidean ring as follows: Let R be a Euclidean ring with Euclidean function ϕ, and assume $t \geq 1$ is a fixed integer. A unitary proper subring D (i.e., $1_D = 1_R$) of R is said to be additively absorptive (or simply, absorptive) of order t in R, if for each f in R there exists g in R such that $f + g \in D$ and $1 \leq \phi(g) \leq t$. Here, we show that this definition is equivalent to the definition of an absorptive subsemiring (given above) for the case of rings. Let D be an absorptive subring of a ring R and f in R. Then there exists g in R such that $f + g$ in D. Let $f + g = h \in D$. then $f = h - g = h + (-g) = h + g'$, where $g' = -g$. Conversely, suppose that there is an element h in D and g in R such that $f = h + g$ with $1 \leq \phi(g) \leq t$. thus, $f + (-g) = h \in D$. Note that in a commutative Euclidean ring, $\phi(a) = \phi(-a)$ for any a in R.*

Example 6. *Let $k[x]$ be the Euclidean ring of polynomials over a field k with the "degree" function as its Euclidean (norm) function (see [3]). Then for any fixed integer $t \geq 1$, the subring $D^{(t)} = \{f \in k[x] \mid x, x^2, \cdots, x^t \text{ coefficients of } f \text{ are zero}\}$ of $k[x]$ is an example of an absorptive subring of order t in $k[x]$.*

Example 7. *Let $k[x]$ be the Euclidean ring of polynomials over a field k with "degree" as its norm, and assume $t \geq 1$ is a fixed odd integer. Define $R^{(t)} = \{f \in k[x] \mid x^j \text{ coefficient(s) of } f \text{ is zero for all odd } j\text{'s with } 1 \leq j \leq t\}$. This is an example of an absorptive subring of order t in $k[x]$.*

Example 8. *Let Z and Q be the rings of integers and rationals, respectively. Then $Z[x]$ is not absorptive of order t in $Q[x]$ for any fixed integer $t \geq 1$. Let $f = 1 + x + x^2 + \cdots + x^t + (1/3)x^{t+1}$. From this, it is clear that $f \notin Z[x]$. Now, it is impossible to have an element $h \in Z[x]$ and $g \in Q[x]$ with $1 \leq \deg(g) \leq t$ such that $f = h + g$.*

Theorem 1. *Let R be a euclidian semiring with euclidian function ϕ, and let $t \geq 1$ be a fixed integer. Assume D_1 and D_2 are proper subsemirings of R with $D_1 \subseteq D_2$. Then the following results are true.*

(i) If D_2 is not absorptive of order t in R, then D_1 is not absorptive of order t in R. Equivalently, if D is an absorptive subsemiring of order t in R, then any proper subsemiring of R that contains D is also absorptive of order t in R.

(ii) let $D_1 \subseteq D_2 \subseteq \cdots$ be an ascending chain of subsemirings in R, then $\cup D_i$ is an absorrptive subsemiring of order t in R provided that $\cup D_i$ is properly contained in R and at least one of the factors in the chain is absorptive of order t in R.

(iii) Let $\{D_i\}$ be a family of proper subsemirings of R. If $\cap_i D_i$ is an absorptive subsemiring of order t in R, then each factor of the intersection is absorptive of order t in R.

Proof. We just give a proof for Part (i) and leave the other two parts to the reader. Assume to the contrary that D_1 is absorptive of order t in R. Let $f \in R_2$. Now, by the assumption, there exists $h \in D_1$ and $g \in R$ such that $f = h + g$ and $1 \leq \phi(g) \leq t$. Hence, we can conclude that D_2 is absorptive of order t in R, which is a contradiction. □

Remark 3. *From Example 8 and Theorem 1, we can conclude that no unitary proper subring of $Z[x]$ can be absorptive of order t in $Q[x]$ for any integer $t \geq 1$.*

3 Main Results

A semiring is said to be zerosumfree whenever $a + b = 0$ implies $a = b = 0$ for all elements a and b in R. A semiring R is (additively) cancellative whenever $a + b = a + c$ implies $b = c$ for all a, b, and c in R. Note that the set N of nonnegative integers is a zerosumfree (resp., cancellative) semiring under the usual addition and multiplication. We will use these facts in the following theorem.

Theorem 2. *Let R be a Euclidean semiring with Euclidean function ϕ satisfying the condition $\phi(ab) = \phi(a) + \phi(b)$ for all elements a and b in R. Then the following results are true.*

(i) $\phi(1_R) = 0$.

(ii) $u \in R$ is a unit in R if and only if $\phi(u) = \phi(1_R) = 0$.

(iii) The identity element 1_R of R is the only nontrivial multiplicative idempotent of R.

Proof. Clearly, for each nonzero element a of R, $\phi(a) = \phi(1_R a) = \phi(1_R) + \phi(a)$ implies that $\phi(1_R) = 0$ since N is additively cancellative. Thus, for each unit $u \in R$, we have $0 = \phi(1_R) = \phi(uu^{-1}) = \phi(u) + \phi(u^{-1})$ which implies $\phi(u) = 0$ since N is a zerosumfree semiring. Now suppose for some $a \neq 0$, $\phi(a) = 0$. Hence, $\phi(a) = \phi(1_R)$ since $\phi(1_R) = 0$. Consequently, the result follows directly from Proposition 11.11 of [2] which states that in a commutative semiring R, $\phi(a) = \phi(1_R)$ if and only if a is a unit. Finally, for the proof of Part (ii), assume $e^2 = e$. Then $\phi(e) = \phi(e^2) = \phi(e) + \phi(e)$ implies $\phi(e) = 0$ since N is cancellative. Now, the result is immediate from Part (ii) since the only unit idempotent in a semiring is its identity element 1. $\square$

A nonempty subset A of a semiring R is subtractive if and only if $a \in A$ and $a + b \in A$ imply $b \in A$. It is strong if and only if $a + b \in A$ implies that $a \in A$ and $b \in A$. Every Subtractive subset of R surely contains 0. Also, it is clear that every strong subset of R is subtractive (see [2]).

Theorem 3. *Let R be a Euclidean semiring with Euclidean function ϕ satisfying the following conditions:*

(1) $\phi(fg) = \phi(f) + \phi(g)$ for all $f, g \in R\backslash\{0\}$ and $fg \neq 0$.

(2) For all $r, g \in R$, if $\phi(r) \leq \phi(g)$, then $\phi(r + g) = \phi(g)$.

Assume $t \geq 1$ is a fixed integer and D is an absorptive and subtractive subsemiring of order t in R. Then for any $f, g \in D$ with $g \neq 0$, there exist $q, r \in D$ such that $f = qg + r$ with $r = 0$ or $\phi(r) < \phi(g)$ or $\phi(r) = i + \phi(g)$ for some $1 \leq i \leq t$.

Proof. Let $f, g \in D$ with $g \neq 0$. Since R is Euclidean, then there exist $q, r \in R$ such that $f = qg + r$ with $r = 0$ or $\phi(r) < \phi(g)$. If $q \in D$, then $r \in D$ since D is subtractive and we are done. Now, suppose $q \notin D$, then by hypothesis, there exists $h \in D$ and $q' \in R$ such that $q = h + q'$ and $1 \leq \phi(q') \leq t$. Consequently, $f = hg + r + q'g$ and since $r + q'g \in D$, by subtractive property of D, it remains only to show that $1 + \phi(g) \leq \phi(r + q'g) \leq t + \phi(g)$. Clearly, $1 \leq \phi(q') \leq t$ implies $1 + \phi(g) \leq \phi(q') + \phi(g) \leq t + \phi(g)$. Thus, from Condition (1), we obtain $1 + \phi(g) \leq \phi(q'g) \leq t + \phi(g)$. Therefore, by applying Condition (2), it follows that $1 + \phi(g) \leq \phi(r + q'g) \leq t + \phi(g)$ since $\phi(r) < \phi(g) \leq \phi(q'g)$. $\square$

Corollary 4. *Let R be a Euclidean semiring with Euclidean function ϕ satisfying the following conditions:*

(1) $\phi(fg) = \phi(f) + \phi(g)$ for all $f, g \in R\backslash\{0\}$ and $fg \neq 0$.

(2) For all $r, g \in R$, if $\phi(r) \leq \phi(g)$, then $\phi(r + g) = \phi(g)$.

Assume $t \geq 1$ is a fixed integer and D is an absorptive and subtractive subsemiring of order t in R. Then for any nonzero proper subsemiring (ideal) I of D and any $f, g \in I$, with $g \neq 0$, there exist $q, r \in D$ such that $f = qg + r$ with $r = 0$, or $\phi(r) < \phi(g)$, or $\phi(r) = i + \phi(g)$ for some $1 \leq i \leq t$.

Theorem 5. *Let R be a Euclidean semiring with Euclidean function ϕ satisfying the following conditions:*

(1) $\phi(fg) = \phi(f) + \phi(g)$ for all $f, g \in R\backslash\{0\}$ and $fg \neq 0$.

(2) For all $r, g \in R$, if $\phi(r) \leq \phi(g)$, then $\phi(r + g) = \phi(g)$.

Assume $t \geq 1$ is a fixed integer and D is an absorptive subsemiring of order t in R. Then for any nonzero proper k-ideal of D and any $f, g \in I$, with $g \neq 0$, there exist $q \in D$ and $r \in I \subseteq D$ such that $f = qg + r$ with $r = 0$, or $\phi(r) < \phi(g)$, or $\phi(r) = i + \phi(g)$ for some $1 \leq i \leq t$.

Proof. Let $f, g \in I$ with $g \neq 0$. Since R is Euclidean, then there exist $q, r \in R$ such that $f = qg + r$ such that $r = 0$ or $\phi(r) < \phi(g)$. If $q \in D$, then $r \in I \subseteq D$ since

I is a k-ideal and we are done. Now, suppose $q \notin D$, then by hypothesis, there exists $h \in D$ and $q' \in R$ such that $q = h + q'$ and $1 \leq \phi(q') \leq t$. Thus, $f = (h + q')g + r = hg + (r + q'g)$. From this and the subtractive property of I, we get $r + q'g \in I$. Hence, it remains only to show that $1 + \phi(g) \leq \phi(r + q'g) \leq t + \phi(g)$. Clearly, $1 \leq \phi(q') \leq t$ implies $1 + \phi(g) \leq \phi(q') + \phi(g) \leq t + \phi(g)$. Now by applying Condition (1), we obtain $1 + \phi(g) \leq \phi(q'g) \leq t + \phi(g)$. Therefore, $1 + \phi(g) \leq \phi(q'g) \leq t + \phi(g)$, and by applying Condition (2), it follows that $1 + \phi(g) \leq \phi(r + q'g) \leq t + \phi(g)$. □

Lemma 6. *Let R be a Euclidean semiring with Euclidean function ϕ satisfying the condition that for all $r, g \in R$, $\phi(r + g) = \phi(g)$ whenever $\phi(r) \leq \phi(g)$. Assume $t \geq 1$ is a fixed integer and D is an absorptive subsemiring of order t in R. Then for each a in R, the condition $\phi(a) = 0$ implies that a must be in D.*

Proof. suppose to the contrary that a is not in D. Since D is absorptive, then there exists $h \in D$ and $b \in R$ such that $a = h + b$ with $1 \leq \phi(b) \leq t$. Thus, $0 = \phi(a) = \phi(h + b) < \phi(b)$. Now, we compare $\phi(h)$ and $\phi(b)$ in three different possible cases and show that it will lead to a contradiction in either case. If $\phi(h) \leq \phi(b)$, then by hypothesis, we will have $1 \leq \phi(b) = \phi(h + b) \leq t$ which is a contradiction. In other case, again by hypothesis, $\phi(h) > \phi(b)$ implies $0 = \phi(a) = \phi(h + b) = \phi(h)$ which is impossible since $\phi(b)$ is strictly larger than zero. We therefore must conclude that if $\phi(a) = 0$, then a must belong to D. □

We shall use this fact in the proof of the first part of the following theorem.

Theorem 7. *Let R be a Euclidean semiring with Euclidean function ϕ satisfying the following conditions:*

(1) $\phi(fg) = \phi(f) + \phi(g)$ for all $f, g \in R \backslash \{0\}$ and $fg \neq 0$.

(2) For all $r, g \in R$, if $\phi(r) \leq \phi(g)$, then $\phi(r + g) = \phi(g)$.

Assume $t \geq 1$ is a fixed integer and D is an absorptive subsemiring of order t in R. Let I be a nonzero proper k-ideal of D with $\phi(I) = inf\{\phi(f) \mid f \in I\} = j$. Then we have the following results:

(i) The ideal I as an ideal of D can be generated by $t + 1$ or fewer elements.

(ii) Assume further that for each nonzero $a \in R$, the condition $1 \leq \phi(a) \leq t$ implies $a \notin D$; and also I contains an element h such that $\phi(h) = i + j$ for some $1 \leq i \leq t$. Then I is not a principal ideal of D.

Proof. Part (i): choose an element g in I with $\phi(g) = j$. By Theorem 5, for each f in I, there exist $q, r \in D$ such that $f = qg + r$ with the following three possibilities: (a) $r = 0$, or (b) $\phi(r) < \phi(g)$, or (c) $\phi(r) = i + \phi(g)$ for some $1 \leq i \leq t$.

Now, the subtractive property of I implies that $r \in I$ and the minimality of $\phi(g)$ disregards the possibility of the case $\phi(r) < \phi(g)$. Also, for those f's in I such that $r = 0$, it is clear that f belongs to (g). Indeed, if for each f in I, Case (a) occurs, then we can conclude that $(g) = I$.

Next, suppose for some element f in I, Case (c) occurs. This implies the existence of an element of I (namely, r in I) with ϕ value equals to $i + \phi(g)$ for some $1 \leq i \leq t$. Thus, $j < \phi(r) \leq t + j$ and clearly $r \in I$ since I, is subtractive. Therefore, it is clear that the set $C = \{m \in N \mid I \text{contains elements that each of which having } \phi \text{ value } m \text{ with } j < m \leq t + j\}$ is not empty. Assume that the cardinality of C is $|C| = k$. Now, label the elements of C as m_1, m_2, $\ldots$, and m_k, where $m_1 > m_2 > \cdots > m_k$. By construction of C, we can choose k elements f_{m_1}, f_{m_2}, $\ldots$, and f_{m_k} from I with $\phi(f_{m_i}) = m_i$, where $i = 1, 2, \ldots, k$. Thus, it is clear that $\phi(r) = \phi(f_{m_{i_1}})$ for some $1 \leq i_1 \leq k$. Since R is a Euclidean ring, then there exist $a_{i_1}, r_1 \in R$ such that $r = a_{i_1} f_{m_{i_1}} + r_1$ with $r_1 = 0$ or $\phi(r_1) < \phi(f_{m_{i_1}})$. Assume that $r_1 = 0$. Then $r = a_{i_1} f_{m_{i_1}}$, and $\phi(r) = \phi(a_{i_1}) + \phi(f_{m_{i_1}})$ implies that $\phi(a_{i_1}) = 0$ (N is cancellative). Therefore, by Lemma 6 (above), $a_{i_1} \in D$. Hence, we obtain $f = qg + r = qg + a_{i_1} f_{m_{i_1}} \in (g, f_{m_{i_1}})$. Now suppose $r_1 \neq 0$ and $\phi(r_1) < \phi(f_{m_{i_1}})$. Thus, we have $\phi(r_1) < \phi(f_{m_{i_1}}) \leq \phi(a_{i_1}) + \phi(f_{m_{i_1}}) = \phi(a_{i_1} f_{m_{i_1}})$, which by hypothesis, it implies $\phi(a_{i_1} f_{m_{i_1}} + r_1) = \phi(a_{i_1} f_{m_{i_1}})$. Now we have $\phi(r) = \phi(a_{i_1} f_{m_{i_1}} + r_1) = \phi(a_{i_1} f_{m_{i_1}}) = \phi(a_{i_1}) + \phi(f_{m_{i_1}})$ which implies $\phi(a_{i_1}) = 0$. Thus, thus again by applying Lemma 6, we get $a_{i_1} \in D$. In this case, it is clear that $\phi(r_1) = \phi(f_{m_{i_2}})$ for some $i_1 < i_2 \leq k$. Again, by the division algorithm, there exist $a_{i_2}, r_2 \in R$ such that $r_1 = a_{i_2} f_{m_{i_2}} + r_2$ with $r_2 = 0$ or $\phi(r_2) < \phi(f_{m_{i_2}})$. By the same argument as we showed $a_{i_1} \in D$, it can be shown that (in either case of $r_2 = 0$ or $\phi(r_2) < \phi(f_{m_{i_2}})$ that $a_{i_2} \in D$. Consequently, whenever $r_2 = 0$, we have $f = qg + r = qg + a_{i_1} f_{m_{i_1}} + r_1 = qg + a_{i_1} f_{m_{i_1}} + a_{i_2} f_{m_{i_2}}$ which clearly belongs to the ideal $(g, f_{m_{i_1}}, f_{m_{i_2}})$. Obviously, by continuing the process of argument as above, we get the elements r, r_1, r_2, $\ldots$ of I with $j + t \geq \phi(r) > \phi(r_1) > \phi(r_2) > \cdots \geq j$. Thus, we reach an element $r_s \in I$ with $\phi(r_s) = \phi(g)$ and $r_{s+1} = 0$. Actually, by the division algorithm in R, there exist $q', r_{s+1} \in R$ such that $r_s = q'g + r_{s+1}$ with $r_{s+1} = 0$ or $\phi(r_{s+1}) < \phi(g)$. But the subtractive property of I implies that $r_{s+1} \in I$,, and therefore, the minimality of $\phi(g)$ excludes the choice of $\phi(r_{s+1}) < \phi(g)$. Hence $r_s = q'g$ and $\phi(r_s) = \phi(q') + \phi(g)$, which implies $\phi(q') = 0$. Now, by Lemma, 6, $q' \in D$. Thus, $f = qg + a_{i_1} f_{m_{i_1}} + a_{i_2} f_{m_{i_2}} + \cdots + q'g$ belongs to $(g, f_{m_1}, f_{m_2}, \ldots, f_{m_k})$, which proves that $I = (g, f_{m_1}, f_{m_2}, \ldots, f_{m_k})$, where $m_1, m_2, \ldots, m_k \in C$. This completes the proof of Part (i).

Finally, for the proof of Part (ii), assume that for each nonzero element a in R, the condition $1 \leq \phi(a) \leq t$ implies $a \notin D$, and also I contains an element h with $\phi(h) = i + \phi(g)$ for some $1 \leq i \leq t$. In this case, suppose to the contrary that I is principal and $I = (g)$ with $\phi(I) = \phi(g) = j$. Hence, $h = qg$ for some q in D. Therefore, $\phi(h) = i + j = \phi(q) + j$ implies $1 \leq \phi(q) = i \leq t$. Thus, by the assumption, this makes $q \notin D$, which is a contradiction. From this, we can conclude that $h \notin (g)$, i.e., I is not contained in (g). In other words, no elements of I with ϕ value equals to j can generate I in D. Now, suppose there exists an element g' in I which generates I in D. From the above argument and minimality of $j = \phi(g)$, we must have $\phi(g') > \phi(g)$. Since $g \in I$, then $g = q'g'$ for some q' in D, and hence, $\phi(g) = \phi(q') + \phi(g') \geq \phi(g')$. This is a contradiction and the proof is complete. □

In view of the above theorem, clearly, every k-ideal of D is a finitely generated ideal in D. Furthermore, in the next section, we will show that D is not a Bezout semiring, which consequently, can never be a Hermite semiring since every Hermite semiring is Bezout.

4 A General Overview

This paper is a natural extention of the work Rahimi [5] on commutative Euclidean rings to commutative Euclidean semirings. In [5], the work of Nick H. Vaughan [6] and [1] has been extended to a special class of Euclidean rings. In [6], it is shown that the subring

$D = \{f \in k[x] \mid x$ coefficient of f is zero$\}$ of $k[x]$ the Euclidean ring of polynomials over a field k satisfying the following conditions:

(1) For any $f, g \in D$, $g \neq 0$, there exist $q, r \in D$ such that $f = qg + r$ with $r = 0$, or deg $(r) <$ deg(g), or deg $(r) = 1 +$ deg (g).
(2) Any ideal I of D can be generated by one or two elements.

In [1, Propositions 3 and 4], Results 1 and 2 (above) are generalized, respectively, to $(1')$ and $(2')$ as follows.
For any fixed integer $t \geq 1$, let $D^{(t)} = \{f \in k[x] \mid x, x^2, \ldots, x^t$ coefficients of f are zero$\}$. Then the following two results are true:

$(1')$ for any $f, g \in D^{(t)}$, $g \neq 0$, there exist $q, r \in D^{(t)}$ such that $f = qg + r$ with $r = 0$, or deg $(r) <$ deg (g), or deg $(r) = i +$ deg (g) for some $1 \leq i \leq t$;
$(2')$ any ideal I in $D^{(t)}$ can be generated by $(t + 1)$ or less elements.

As we mentioned above, the main purpose of [5] paper was a natural extension of Propositions 3 and 4 in [1] for unitary subrings of a special class of Euclidean rings. A unitary subring D of a unitary ring (i.e., a ring with identity) R is a subring of R with $1_D = 1_R$.
In this section we mention some properties of $D^{(t)}$ as a general form of $D^{(1)}$ which was studied in [6, Sec. 3]. It is fairly routine to show that $D^{(t)}$ satisfies the properties of D $(=D^{(1)})$ as stated in [6, Sec. 3]. Additionally, since $D^{(t)}$ is not integrally closed, therefore, it is clearly neither Prufer nor Dedekind. Similarly, $D^{(t)}$ is furthermore not a valuation domain [4, pp. 12-14], nor a pseudo-Bezout domain [4, p. 15] (a domain is pseudo-Bezout if every pair of elements has a greatest common divisor). Thus, $D^{(t)}$ is also not Bezout.

Remark 4. *In the statement of Theorem 7, we assume $\phi(r + g) = \phi(g)$ provided that $\phi(r) \leq \phi(g)$. This assumption is very natural and exactly similar to the case for $k[x]$ of the Euclidean ring of polynomials over a field k with the "degree" function as its Euclidean norm (function). But for the same condition in Theorem 3 of [5], we assumed that $\phi(r+g) = \phi(g)$ whenever $\phi(r) < \phi(g)$. Note that in the proof of the first part of Theorem 3 of [5], we have used the assumption that for any a in R with $1 \leq \phi(a) \leq t$, then $a \notin D$. But in this paper, for the proof of Part (i) of Theorem 7, we indeed by virtue of Lemma 6 have relaxed this condition. Actually, the proof of part of this lemma depends on the assumption that "$\phi(r + g) = \phi(g)$ whenever $\phi(r) \leq \phi(g)$" rather than the condition "$\phi(r + g) = \phi(g)$ whenever $\phi(r) < \phi(g)$" which was used in [5] and was not applicable to prove a lemma similar to Lemma 6.*

A semiring R is said to be Hermite provided that for all $a, b \in R$, there exist $a_1, b_1, d \in R$ such that $a = a_1d$, $b = b_1d$, and $(a_1, b_1) = R$. In the following lemma, we show that a Hermite semiring is necessarily Bezout. A Bezout semiring is a semiring in which every finitely generated ideal is principal.

Lemma 8. *Every Hermite semiring is a bezout semiring.*

Proof. Proof by induction. Let $n \geq 2$ be a fixed integer and $(x_1, x_2, \cdots, x_n)$ an ideal in a semiring R. By the hypothesis, for x_1 and x_2, there exist a_1, a_2, and d_1 in R such that $x_1 = a_1d_1$, $x_2 = a_2d_1$, and $(a_1, a_2) = R$. From this, it is clear that the ideal $(x_1, x_2) \subseteq (d_1)$. On the other hand, $1 = r_1a_1 + r_2a_2$ for some r_1 and r_2 in R. Hence, $d_1 = r_1a_1d_1 + r_2a_2d_1$ which implies $d_1 \in (x_1, x_2)$. Consequently, $(x_1, x_2) = (d_1)$, which implies $(x_1, x_2, x_3) = (d_1, x_3) = (d_2)$ for some d_2 in R. Now by an inductive argument, we get $(x_1, x_2, \cdots, x_{n-1}, x_n) = (d_{n-2}, x_n) = (d_{n-1})$ for some d_{n-1} in R. $\square$

Theorem 9. *Let R be a semiring that satisfies Part (ii) of Theorem 7 (above). Then R is not bezout, and consequently, not a Hermite semiring.*

Proof. The proof follows directly from Part (ii) of Theorem 7 and the above lemma. $\square$

References

[1] Chapman, S.T. and Vaughn, N.H., *A theorem on generating ideals in certain semigroup rings*, Bollettino U.M.I., (7) 5-A (1991), 41-49.

[2] Golan, J.S., *The theory of semirings (with applications in mathematics and theoretical computer science)*, Pitman Monographs & Surveys in Pure and Appl. Math., vol. 54, Longman Scientific & Tech., Essex, 1992.

[3] Hungerford, T.W., *Algebra*, Springer-Verlag New York Inc. 1974.

[4] Hutchins, H.C., *Examples of commutative rings*, Polygonal Publishing House, NJ, 1981.

[5] Rahimi, A.M., *Rings with an almost division algorithm*, Libertas Mathematica, vol. 13 (1993) 41-46.

[6] Vaughan, N.H., *An integral domain with an almost division algorithm*, Journal of Natural Sciences and Mathematics, 21 (1), (1981) 81-84.