LIBERTAS MATHEMATICA, VOL. XV (1995)

ON GEL'FAND ELEMENTS OF BANACH ALGEBRAS AND RELATED PROBLEMS OF HARMONIC ANALYSIS

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ABSTRACT : We introduce in this paper the notion of a Gel'fand element in a Banach algebra with involution. When this algebra possesses sufficientely many irreducible *-representations, we give a characterization by means of Representation theory of such elements. We determine Gel'fand elements for a class of C^*-subalgebras of the algebra of all bounded operators in a Hilbert space. We give some sufficient conditions of existence of Gel'fand elements in a general C^*-algebra using the notion of Moore-Penrose inverse. In the case of measure algebra of a group, Gel'fand elements are called Gel'fand measures and were introduced in [Ak-Ba]. We determine the Gelfand measures of the measure algebra of a finite group. Finaly, in the case of Gel'fand measure μ with a compact support in a unimodular group G with a growth being atmost of polynomial type, we deal with a specialized question of Harmonic Analysis namely, we prove that the algebra $L_1^\mu(G) := \mu * L_1(G) * \mu$ is regular, symmetric and satisfies Wiener condition : that is, the ideal $\mathcal{I}^\mu$ formed by all elements in $L_1^\mu(G)$ having their μ-spherical Fourier transform of compact support in the set of characters of the commutative algebra $L_1^\mu(G)$ (cf.[Ak-Ba]).

Kew words : Gel'fand element, measure algebra, Gel'fand measure, Wiener property. C^*-algebra. Moore-penrose inverse. *-representations. *-Banach algebras. Symmetric algebra. Regular algebra...

A.M.S. S.C. : 43 A 05 ; 43 A 10 ; 43 A 30.

INTRODUCTION : In the section $n^0 1$, we consider a banach algebra $\mathcal{A}$ with an involution denoted by * and a unit element denoted by 1. we say that an element $e \in \mathcal{A}$ is a Gel'fand element if e is idempotent (i.e. $e^2 = e$), hermitian (i.e. $e^* = e$ and the closed subalgebra $e\mathcal{A}e$ is commutative. In the theorem 1.1., when the algebra $\mathcal{A}$ has sufficientely many irreducible *-representations, we give a characterization by means of Representation theory for its Gel'fand elements. For a given Hilbert space $\mathcal{H}$, denote by $\mathcal{A}$ a C^*-subalgebra of $\mathcal{L}(\mathcal{H})$, the algebra of all bounded operators in $\mathcal{H}$, if $\mathcal{A}$ is irreducible and contains a compact perator, then an operator E is a Gel'fand element in $\mathcal{A}$ if and only if E is a projection of rank one. In general, it may hapen that a C^*-algebra fails to contain Gel'fand elements(cf.[Co]). The existence in a C^*-algebra of normal, regular and abelian elements implies the existence of Gel'fand elements. Consider $\mathcal{B}$ a subalgebra of $\mathcal{A}$ and let e be a Gel'fand element in $\mathcal{B}$: we give in theorem 1.4., necessary and sufficient conditions for e to be a Gel'fand element in $\mathcal{A}$. We apply theorem 1.1. in the section $n^0 2$, to describe Gel'fand measures of a finite group. The result obtained is given in the theorem 2.1. In third and last section, we consider the case of a unimodular group with growth being atmost of polynomial type. We prove, in theorem 3.5., by using a functionnal calculus introduced by J.Dixmier (cf.[Di]) and used by O.Gebuhrer (cf.[Ge]) in his work on Gel'fand - Levitan spaces; that if μ is a Gel'fand measure such that its support is compact in G, then the algebra $L_1^\mu(G) := \mu * L_1(G) * \mu$ satisfies the Wiener property : that is, the ideal $\mathcal{I}^\mu(G)$ formed by all elements in $L_1^\mu(G)$ having their μ-spherical Fourier transform of compact support in the set of characters of the commutative algebra $L_1^\mu(G)$ (cf.[Ak-Ba]). In particular, this is true if the group G is commutative or compact.

1.Gel'fand elements in a Banach algebra with involution

In this section, $\mathcal{A}$ means a complex Banach algebra with unit element 1 and an involution denoted by * . We assume that $\mathcal{A}$ has a complete set of irreducible*- representations. This is realized if,for example,$\mathcal{A}$ is reduced.We shall denote $\widehat{\mathcal{A}}$ the set of all *- irreducible representations of $\mathcal{A}$.

Definition 1.1. Let $e \in \mathcal{A}$ satisfying $e^2 = e^* = e$. we say that e is a Gel'fand element in $\mathcal{A}$ if the closed subalgebra $e\mathcal{A}e$ is commutative.

The following theorem gives characterisations of Gelfand elements in$\mathcal{A}$ by use of representation theory.

THEOREM 1.2. *let $\mathcal{A}$ be an involutive Banach algebra having a unit element and a complete set of irreducible *-representations. Let $e \in \mathcal{A}$ verifying $e = e^2 = e^*$. Then the following are equivalents :*

(1) e is a Gel'fand element in $\mathcal{A}$.
(2)For all $(\pi, \mathcal{H}_\pi) \in \hat{\mathcal{A}}$; the rank of the operator $\pi(e)$ is 0 or 1 .
(3)For all $(\pi, \mathcal{H}_\pi) \in \hat{\mathcal{A}}$ and for all $x \in e\mathcal{A}e$ the operator $\pi(x)$ is of rank 0 or 1.
(4)For all $(\pi, \mathcal{H}_\pi) \in \hat{\mathcal{A}}$ the dimension of the space $\{\pi(x) : x \in e\mathcal{A}e\}$ is 0 or 1 .

Proof :

(1) $\Longrightarrow$ (2) : Suppose that $e\mathcal{A}e$ is commutative. Let $(\pi, \mathcal{H}_\pi)$ be an irreducible *-representation of $\mathcal{A}$. For all vectors ξ, η in the space $\mathcal{H}_\pi$; we consider the operator $E_{\xi,\eta}$ defined by :

$$E_{\xi,\eta}(\theta) := < \theta \mid \eta > \xi;$$

for all θ in the space $\mathcal{H}_\pi$; where $<|>$ means the scalar product in this space.

For all elements x, y in $\mathcal{A}$, we have :

$$\pi(e)\pi(x)\pi(e)\pi(y)\pi(e) = \pi(e)\pi(y)\pi(e)\pi(x)\pi(e).$$

By assumption, π is irreducible. Then Von-Neumann density theorem (cf. [Ga]) ensures that the subspace $\pi(\mathcal{A})$ is strongly dense in the algebra $\mathcal{L}_\infty(\mathcal{H}_\pi)$ of all bounded linear operators on the hilbert space $\mathcal{H}_\pi$. Then the following identity holds for all bounded operators S and T on the space $\mathcal{H}_\pi$:

$$\pi(e)S\pi(e)T\pi(e) = \pi(e)T\pi(e)S\pi(e).$$

If we take $S = E_{\xi,\eta}$ and $T = E_{\eta,\xi}$ where ξ, η in the space $\mathcal{H}_\pi$; then we have :

$$\|\pi(e)\eta\|^2 E_{\pi(e)\xi,\pi(e)\xi} = \|\pi(e)\xi\|^2 E_{\pi(e)\eta,\pi(e)\eta}$$

Which proves that $\pi(e)$ is of rank 0 or 1.

(2) $\Longrightarrow$ (3) Is evident.

(3) $\Longrightarrow$ (4) It follows from the fact that : $\{\pi(x); x \in e\mathcal{A}e\}$ is included in the space of dimension one : $\mathbb{C}\pi(e)$.

(4) $\Longrightarrow$ (1) results from the assumption made on $\mathcal{A}$ to have a complete set of irreducible *-representations. $\square$

REMARKS : (a) If G is alocally compact group and $\mathcal{A} = M_1(G)$ is the Banach algebra of bounded complex measures on G. A Gel'fand element of $\mathcal{A}$ is said a Gel'fand measure (cf. [Ak,Ba]) .

(b) The notion of Gel'fand element may be defined- in general-in an involutive algebra having or not a unit element.The theorem 1 is valid in an *-Banach algebra $\mathcal{A}$ which has a complete set of irreducible *-representations and just an approximation of unit.

(c) Let $\mathcal{A}$ be an *-Banach algebra with unit element. Let B be an involutive subalgebra (without unit element) of $\mathcal{A}$ dense in $\mathcal{A}$ or just having an approximation of unit. Let $e \in \mathcal{A}$, such that : $e^2 = e^* = e$, then e is a Gel'fand element in $\mathcal{A}$ if and only if eBe is commutative.

NOTATIONS : Let $\mathcal{A}$ be a Banach *-algebra, and $e \in \mathcal{A}$ satisfying : $e^2 = e^* = e$. For all *-representation $(\pi, \mathcal{H})$ of $\mathcal{A}$, we will denote $\mathcal{H}_e$ the closed subspace invariant by π and generated by the range of the operator $\pi(e)$. For all $x \in \mathcal{A}$, the restriction of the operator $\pi(x)$ to the subspace $\mathcal{H}_e$ (resp. to the orthogonal : $\mathcal{H}_e^\perp$ of the space $\mathcal{H}_e$) will be denoted : $\pi_e(x)$ (respect. $\pi_e^\perp(x)$) . We obtain then two *-representations of $\mathcal{A}$: $(\pi_e, \mathcal{H}_e)$ and $(\pi_e^\perp, \mathcal{H}_e^\perp)$ verifying :

$$\pi = \pi_e \oplus \pi_e^\perp; \qquad \pi_e^\perp(e) = 0;$$

and $\pi_e(e)$ is the restriction of π to $\mathcal{H}_e$. In particular, the operators $\pi(e)$ and $\pi_e(e)$ have the same range.

Let $\hat{\mathcal{A}}_e$ denote the set of all irreducible *-representations $(\pi, \mathcal{H})$ of $\mathcal{A}$ such that : $\pi(e) \neq 0$.

PROPOSITION 1.3. *Let $e \in \mathcal{A}$ such that $e^2 = e^* = e$. Let $(\pi, \mathcal{H})$ be an *-representation of $\mathcal{A}$, with a cyclic vector ξ which is e-invariant (i.e. $\pi(e)\xi = \xi$) . If $\pi(e)$ is of rank one , then the representation $(\pi, \mathcal{H})$ is irreducible.*

Proof : Let us denote $\mathcal{H}_0$ the closed subspace of $\mathcal{H}$ wich is invariant by π. Then the orthogonal projection P_0 on $\mathcal{H}_0$ commutes with $\pi(e)$. Write $\xi_0 = P_0(\xi)$. By assumption, we must have : $\xi_0 = \alpha\xi$, where α is a complexe number. then, two cases occur : first case, if $\alpha \neq 0$, then $\xi \in \mathcal{H}_0$, wich implies, since ξ is cyclic, that $\mathcal{H}_0 = \mathcal{H}$. Second case, If $\alpha = 0$, then ξ belongs to $\mathcal{H}_0^\perp$ the orthogonal of $\mathcal{H}_0$, since this space is itself invariant by π, we deduce that $\mathcal{H}_0 = 0$. □

THEOREM 1.4. *Let $\mathcal{A}$ be a Banach *-algebra. Let $\mathcal{B}$ a banach *-subalgebra of $\mathcal{A}$. Consider $e \in \mathcal{B}$ a Gel'fand element of $\mathcal{B}$. Then the following statements are equivalents :*

(1) e is a Gel'fand element of $\mathcal{A}$.
(2) For all *-representation $(\pi, \mathcal{H}) \in \hat{\mathcal{A}}_e$ we have :
$(\pi|_{\mathcal{B}})_e$ is irreducible; where $\pi|_{\mathcal{B}}$ is the restriction of π to $\mathcal{B}$.

Proof :

(1) $\Longrightarrow$ (2). let $(\pi, \mathcal{H})$ be an irreducible *-representation of $\mathcal{A}$ verifying $\pi(e) \neq 0$.Then there exists an element $\xi \in \mathcal{H}$, such that $\|\xi\| = 1$ and $\pi(e) = E_{\xi,\xi}$. Let ρ denote the restriction of π to the subalgebra $\mathcal{B}$. Then we have :

$$\rho = \rho_e \oplus \rho_e^{\perp} \text{ and } \mathcal{H} = \mathcal{H}_e \oplus \mathcal{H}_e^{\perp}$$

where $\mathcal{H}_e$ is the closed subspace of $\mathcal{H}$ invariant by $\{\rho(h) : h \in \mathcal{B}\}$ generated by the range of $\rho(e)$ which coincides with $\mathbb{C}\xi$, the range of $\pi(e)$. This implies that $\xi \in \mathcal{H}_e$; and $\rho_e(e)\xi = \pi(e)\xi = \xi$.

Then ξ is a cyclic vector for the representation $(\rho_e, \mathcal{H}_e)$. The proposition 1.3 ensures that $(\rho_e, \mathcal{H}_e)$ is an irreducible representation of $\mathcal{B}$ and belongs to $\hat{\mathcal{B}}_e$.

(2) $\Longrightarrow$ (1). Let $(\pi, \mathcal{H}) \in \hat{\mathcal{A}}_e$. Let ρ be the restriction of π to the Banach *-subalgebra $\mathcal{B}$. By assumption, $\rho_e \in \hat{\mathcal{B}}$, since $\pi(e) \neq 0$ then $\rho_e(e)$ is of rank one. Hence,the operator $\pi(e)$ is of rank one in the space $\mathcal{H}$. []

2. Gel'fand elements of C^*-algebras .

2.1. case of C^*-algebras of bounded operators in a Hilbert space .

Consider $\mathcal{L}(\mathcal{H})$ the algebra of all bounded operators in a Hilbert space $\mathcal{H}$. Recall that for all vectors ξ, η in $\mathcal{H}$, $E_{\xi,\eta}$ means the operator of rank one defined by :

$$E_{\xi,\eta}(\theta) := < \theta \mid \eta > \xi;$$

for all θ in the space $\mathcal{H}_\pi$; where $<|>$ means the scalar product in this space. We denote $\mathcal{L}_c(\mathcal{H})$ the closed two-sided ideal in $\mathcal{L}(\mathcal{H})$ formed by all compact operators. The following result gives a characterisation of Gelf'and elements for a class of C^*-subalgebras of $\mathcal{L}(\mathcal{H})$.

PROPOSITION 2.1.1. *Let $\mathcal{A}$ be an irreducible C^*-algebra contained in $\mathcal{L}(\mathcal{H})$ such that $\mathcal{A} \cap \mathcal{L}_c(\mathcal{H}) \neq 0$. Let E be in $\mathcal{A}$. Then E is a Gel'fand element in $\mathcal{A}$ if and only if E is of the type $E_{\xi,\xi}$ where $\xi \in \mathcal{H}$ is a vector with norm one.*

Proof : *The assumptions made on $\mathcal{A}$ implie that $\mathcal{L}_c(\mathcal{H})$ is contained in $\mathcal{A}$. (see for example [Do], theorem 5.39, p.141).Now let E be a Gel'fand element in the algebra $\mathcal{A}$. Then E satisfies :*

$$ETESE = ESETE \qquad \forall T, S \in \mathcal{A} \tag{1}$$

Since $E \neq 0$, then there exists $\eta \in \mathcal{H}$ such that $< E\eta \mid \eta > \neq 0$. Now fix such η and take $\zeta \in \mathcal{H}$. By (1), we have :

$$EE_{\zeta,\eta}EE_{\eta,\zeta}E = EE_{\eta,\zeta}EE_{\zeta,\eta}E$$
$$< E\eta \mid \eta > E_{E\zeta,E^*\eta} = < E\zeta \mid \zeta > E_{E\eta,E^*\eta}$$

Wich proves that $E\zeta$ and $E\eta$ are proportionnals. Hence the operator E is an orthogonal projection of rank one.

The converse is obvious. □

As a consequence, we have :

COROLLARY 2.1.2. *Let E be in $\mathcal{L}(\mathcal{H})$. Then : E is a Gel'fand element in $\mathcal{L}(\mathcal{H})$ if and only if E is of the type $E_{\xi,\xi}$ where $\xi \in \mathcal{H}$ is a vector with norm one.*

2.2. General case :

In general a C^*-algebra does not have Gel'fand elements. In the paper [Co] J.M.Cohen gave example of C^*-algebras without idempotents. We give here some sufficients conditions for existence of Gel'fand elements in a given C^*-algebras.

PROPOSITION 2.2.1. *Let $\mathcal{A}$ be a (non commutative) C^*-algebra having a unit element. If $\mathcal{A}$ contains a normal regular element $a \in \mathcal{A}$ such that $a\mathcal{A}a$ is commutative, then $\mathcal{A}$ contains Gel'fand elements.*

Proof : Since a is regular (i.e. $a \in a\mathcal{A}a$), then by a result from [Ha-Mb] the element aa^* is also regular. Furthermore, the closed subalgebra $aa^*\mathcal{A}aa^*$ is commutative. So, without loss of generality we can suppose that a is regular and selfadjoint.

By ([Ha-Mb], theorem 6) a must have a (unique) Moore-Penrose inverse $a^+ \in \mathcal{A}$. The element a^+ is also selfadjont and by definition commutes with a. Put $e := aa^+$. Then e is a Gel'fand element in $\mathcal{A}$. □

3. Gel'fand elements of measure algebras of groups.

Let G be a locally compact group. we mean by $\mathcal{M}_1(G)$, the algebra of all complex bounded measures on G. We denote $\mathcal{K}(G)$ the space of continuous functions on G with compact support. We call Gel'fand measure (cf.[Ak] ;[Ak-Ba]) a measure which is a Gel'fand element of $\mathcal{M}_1(G)$. In the first subsection we study the finite case .

3.1. Gel'fand measures of a finite group.

Let G be a finite group and denote $|G|$ its order. Recall that in this case $\mathcal{M}_1(G)$ coincides with the algebra of all complex functions on G and that $\hat{G}$ is finite. put $\hat{G} = \{\lambda_1, ..., \lambda_N\}$; where $N = |\hat{G}| \in \mathbb{N}^*$. For $j = 1, 2, ..., N$ we denote $d_j = \dim \mathcal{H}_j$, where $(\pi_j, \mathcal{H}_j)$ is an element of the class λ_j and for $\xi \in \mathcal{H}_j$ with $\|\xi\| = 1$ the function :

$$C_j(t) := < \xi | \pi_j(t)\xi >; \ t \in G$$

is called a coefficient associated to λ_j for $j = 1, 2, ..., N$.

With these notations, we remark that if μ is a function of type :

$$\mu(t) := \sum_{\lambda \in S} d_\lambda C_\lambda(t) ;$$

where $S \subset \hat{G}$ and C_λ is a a coefficient associated to λ . then μ is a Gel'fand measure on G Conversely, we have :

THEOREM 3.1.1. *For all Gel'fand measure μ on G, there exists a unique subset $S \subset \hat{G}$ such that :*

$$\mu := \sum_{\lambda \in S} d_\lambda C_\lambda ;$$

whre C_λ is a coefficient associated to λ .

Proof : Let μ be a Gel'fand measure on G, the theorem of Peter-Weyl shows that we can write :

$$\mu(t) = \sum_{j=1}^{N} d_j \sum_{k,l=1}^{d_j} \alpha^j_{k,l} < e^j_k | \pi_j(t) e^j_l > ;$$

where $(\pi_j, \mathcal{H}_j)$ is an element of the class λ_j and $\{e^j_l : 1 \leq d_j\}$ is an orthonormal basis of the space $\mathcal{H}_j$; for $j = 1, ..., N$.

Put : $S := \{\pi_\lambda(\mu) \neq 0 ; \pi_\lambda \in \lambda\}$. Hence $\pi_\lambda(\mu)$ is of rank one for all $\lambda \in S$. (If S is empty, then $\mu = 0$. Trivial case.) Let $\lambda \in S$, take $k_0 \in \{1, ..., N\}$ and consider $(\pi_{k_0}, \mathcal{H}_{k_0}) \in \lambda$ (i.e. $\lambda = \lambda_{k_0}$). Then a calculation using Schur orthogonality relations proves that :

$$< \pi_{k_0}(\mu) e^{k_0}_m | e^{k_0}_n > = \alpha^{k_0}_{n,m} .$$

Since the operator $\pi_{k_0}(\mu)$ is of rank one, then we can construct an orthonormal basis $\{\eta_1^{k_0},...,\eta_{d_{k_0}}^{k}\}$ of the space $\mathcal{H}_{k_0}$ such that the first vector $\eta_1^{k_0}$ generates the range of the operator $\pi_{k_0}(\mu)$. these considerations show that in suitable bases we can write μ as :

$$\mu := \sum_{\lambda \in S} d_\lambda C_\lambda . \ \square$$

3.2. On the existence of Gel'fand measures on groups and other problems :

In the papers [Ak] and [Ak-Ba], many examples of Gel'fand measures on groups are given, a theory is established for them generalising the notion of Gel'fand pairs (cf.[Fa]). But the study of their sructure, nature of their support,a general theorem of existence,... are not examinated yet. Note that, contrary to the case of Gel'fand pairs where the group must be unimodular, our theory is valid even in the non unimodular case, that is : there exist non unimodular groups having Gel'fand measures (cf.[Ak-Ba] and [Ak],for examples).

3.3. Wiener property for algebras associated to Gel'fand measures with compact support on groups of polynomial growth :

In all this subsection, we assume that the group G is unimodular its growth being atmost of polynomial type. Let μ be a Gel'fand measure on G such that its support is compact in G. Call e the unit element of G, σ the Haar measure of G. Recall that the Plancherel measure ω associated to μ (cf.[Ak],[Ak-Ba]) has the set $\hat{G}_\mu := \{\pi \in \hat{G} : \pi(\mu) \neq 0\}$, as support.

Firstly, let us introduce the functionnal calculus of J.Dixmier (cf.[Di]) to the *-Banach commutative algebra $L_1^\mu(G) := \mu * L_1(G) * \mu$:

Let $N \in \mathbb{N}$ be a fixed integer majorizing at infinite the growth of G that is : $\sigma(V^n) = O(n^N)$, when $n \longrightarrow \infty$, where V is a compact neighborhood of e. For all complex number λ and $f \in L_1^\mu(G)$, we denote $exp^* i\lambda f$ the element of $\mathbb{C}\delta_e \oplus L_1^\mu(G)$. ($\delta_e$ being Dirac measure concentrated at e) defined by :

$$exp^*(i\lambda f) := \sum_{n=0}^{\infty} \frac{(i\lambda)^n}{n!} f^{*n} ;$$

where $f^{*n} := f * f ... * f$ (n times).

LEMMA 3.3.1. *Let $f \in L_1^\mu(G) \cap L_2^\mu(G)$ with compact support and verifying $\tilde{f}(x) = f$, where $\tilde{f}(x) := \overline{f(x^{-1})}$; $x \in G$. Let ϕ be a complex function defined on* ℝ *(set of the reals) having all derivatives of order $\leq N+3$ continuous and integrables with $\phi(0) = 0$. Then :*

(1) the integral : $\frac{1}{2\pi}\int_{\mathbb{R}} exp^*(i\lambda f)\hat{\phi}(\lambda)d\lambda$ converges absolutly in $L_1^\mu(G)$ to an element denoted by $\phi\{f\}$.
(2) For all hermitian character χ of the Banach algebra $L_1^\mu(G)$, we have $\widehat{\phi\{f\}}(\chi) = \phi(\hat{f}(\chi))$; where $\widehat{\phi\{f\}}$ is the μ–spherical Fourier transform (cf.[Ak-Ba]) of the function $\phi\{f\}$ belonging to the algebra $L_1^\mu(G)$.
(3) If $\phi(t) = t^p$ $(p \in \mathbb{N})$; for $|t| \leq \|f\|_1$; then $\widehat{\phi\{f\}} = f^{*p}$.

One can prove this lemma by adapting (as in [Ge]) the proof of J.Dixmier (cf.[Di]).

THEOREM 3.3.2. *The *-Banach commutative algebra $L_1^\mu(G)$ is symmetric.*

The proof of this theorem is analogous to one given in [Vo] in the context of Hypergroups.

THEOREM 3.3.3. *The *-Banach commutative algebra $L_1^\mu(G)$ is regular.*

Proof : It is sufficient to prove that if $\chi_0 \in \Sigma_\mu$ (the Gel'fand spectrum of $L_1^\mu(G)$) is a character and Φ is a closed subset of the set of all characters of $L_1^\mu(G)$ such that $\chi_0 \notin \Phi$; then there exists a function $g \in L_1^\mu(G)$ verifying : $\hat{g}(\chi_0) = 1$ and $\hat{g}(\chi) = 0$ for all $\chi \in \Phi$.

Let α be a continuous function with compact support on the set of all characters of $L_1^\mu(G)$ satisfying :

$$\alpha(\chi_0) = 1; \qquad 0 \leq \alpha \leq 1 \qquad \text{and} \qquad \text{supp}(\alpha) \subset \Phi^c.$$

Since the set $\mathcal{F}(L_1^\mu(G))$ is dense in the space $C_0(\Sigma_\mu)$, where $\mathcal{F}$ is the μ-spherical Fourier transform ; then we can find a function $g \in L_1^\mu(G)$, hermitian and satisfying $\|\hat{g} - \alpha\|_\infty < \frac{1}{4}$. The density of the subalgebra $\mu * \mathcal{K}(G) * \mu$ in the algebra $L_1^\mu(G)$ ensures the existence of a hermitian function $f \in L_1^\mu(G)$ with compact support such that : $\|f - g\|_1 < \frac{1}{4}$.

One can verify that : $\left|\hat{f}(\chi_0)\right| > \frac{1}{2}$. This shows that we may choose $\phi \in \mathcal{S}(\mathbb{R})$ ($\mathcal{S}(\mathbb{R})$ being the Shartz space) verifying :

$$\widehat{\phi\{f\}}(\chi_0) = 1 \text{ and } \text{supp}(\phi) \subset \mathbb{R} \setminus [-\frac{1}{2}, \frac{1}{2}].$$

One can verify also that : $|\hat{f}(\chi)| < \frac{1}{2}$ for all $\chi \in \Phi$. This implies that the function $\phi\{f\}$ given by lemma 3.3.1. satisfies :

$$\widehat{\phi\{f\}}(\chi_0) = 1 \qquad \text{and} \qquad \widehat{\phi\{f\}}(\chi) = 0; \text{ forall } \chi \in \Phi.$$

This completes the proof. □

We shall denote by $\mathcal{I}_1^\mu(G)$ the ideal formed by all functions $f \in L_1^\mu(G)$ such that $\mathcal{F}(f)$ has compact support in the space Σ_μ.

The following lemma is due to J.Dixmier (cf.[Di]).

LEMMA 3.3.4. *Let r be an integer ≥ 1. Let $\phi_0 \in \mathcal{S}(\mathbb{R})$, such that : $\phi_0(t) = t^{r+1}$ in a neighbourhood of 0. Then for all $\epsilon > 0$, there exists a function $\phi \in \mathcal{S}(\mathbb{R})$, verifying :*

(1) $\phi(t) = \phi_0(t)$ if $|t| \geq 1$;
(2) $\phi(t) = 0$ in a neighbourhood of 0 and
(3) $\sup\{\left\|\phi^{(j)} - \phi_0^{(j)}\right\|_\infty ;\ j = 0, ..., r\} \leq \epsilon$.

Now, applying the functionnal calculus introduced before, we are able to prove Wiener property for the algebra $L_1^\mu(G)$ precisely, we have :

THEOREM 3.3.5. *The ideal $\mathcal{I}_1^\mu(G)$ is dense in $L_1^\mu(G)$.*

proof :

Let N be an integer dominating the growth of G at infinite . Let K denote the support of μ. K is a compact subgroup of G. Let V be a neighbourhood of 1, the unit element of G. Then there exists a compact neighbourhood U of 1, such that $U \subset V$.

Let f be a hermitian and positive element of $\mathcal{K}(G)$ having a compact support contained in U with integral being equal to one . Let $\phi_0 \in \mathcal{S}(\mathbb{R})$ be a function verifying $\phi_0(t) = t^{N+4}$ for $|t| \leq \|f^\mu\|_1$. From lemma 3.3.1 , we deduce that : $\phi_0\{f^\mu\} = [f^\mu]^{*(N+4)}$. Then $\phi_0\{f^\mu\}$ is an element of $L_1^\mu(G)$ .The lemma 3.3.4. , proves the existence of a sequence $\{\phi_n\}$ formed by elements $\phi_n \in \mathcal{S}(\mathbb{R})$ such that $\phi_n\{f^\mu\}$ converges in $L_1(G)$ to $\phi_0\{f^\mu\}$. We kow by the same lemma that for all $n \geq 0$, the function ϕ_n vanishes on a neighbourhood of 0,as a consequence, we deduce that $\phi_0\{f\}$ belongs to the closure of $\mathcal{I}_1^\mu(G)$ in the space $L_1^\mu(G)$.

Let $g \in L_1^{\mu}(G)$. Let $\epsilon > 0$.Then there exists V be a neighbourhood of 1 such that for all positive $h \in \mathcal{K}(G)$ with integral being equal to one and support included in V, we have : $\|g - g * h\|_1 \leq \epsilon$.

Now, take f and ϕ_0 as above, then by applying :

$$\|f^{\mu}\|_1 \leq \|\mu\|^2 \qquad \text{and} \qquad \|g - g * f^{\mu}\|_1 \leq \epsilon\|\mu\|^2.$$

we have the following inequalities :

$$\|g - g * \phi_0\{f^{\mu}\}\|_1 = \left\|g - g * [f^{\mu}]^{*(N+4)}\right\|_1$$

$$\leq \sum_{j=1}^{N+4} \left\|g * [f^{\mu}]^{*j-1} - g * [f^{\mu}]^{*j}\right\|_1$$

$$\leq \|g - g * f^{\mu}\|_1 \sum_{j=1}^{N+4} \|\mu\|^{2(j-1)} \qquad \leq M\epsilon.$$

Where M is a positive constant depending only on N and μ. □

REMARKS : (a) As a consequence, we deduce that for all μ-spherical function ϕ on G, we have $\phi(x^{-1}) = \overline{\phi(x)}$.

(b) If we take G commutative and μ the Dirac measure, then we find a classical result of harmonic analysis on locally compact abelian groups (see for example [Re],[He-Ro],...).

REFERENCES

[Ak] M. AKKOUCHI, *Mesures de Gel'fand et fonctions sphériques généraliseés.* Thèse d'etat. Univ. Ibn Tofail, Kenitra (Maroc) 1994.

[Ak-Ba] M. AKKOUCHI AND A. BAKALI, une généralisation des paires de Gel'fand, *Bollettino U.M.I.* (7) 6-B (1992), 795-822.

[Co] J.M. COHEN, C^*-algebras without idempotents, *Jour. of Funct. Analysis.* 33 (2) (1979), 211-216.

[Di] J.. DIXMIER, *Opérateurs de rang fini dans les representations unitaires,* I.H.E.S.S. (1960), 305-317.

[**Do**] R.G. DOUGLAS, *Banach algebras techniques in operator theory*, Academic Press, new York .

[**Fa**] J. FARAUT, *Analyse harmonique sur les espaces Riemanniens symétriques de rang un*, cours C.I.M.P.A., Université de Nancy 1, (1980).

[**Ga**] S.A. GAAL, *Linear Analysis and representation theory*. Springer-Verlag, Berlin, 1973.

[**Ge**] O. GEBUHRER, *Analyse harmonique sur les espaces de Gel'fand- Levitan et applications à la theorie des semi-groupes de convolution*. Thèse d'etat. Univ. L.Pasteur, Strasbourg (France) 1989.

[**Ha-Mb**] R. HARTE AND M. MBEKHTA, on generalized inverses in C^*−algebras, *Studia mathematica*. 103 (1) (1992), 71-77.

[**He-Ro**] E. HEWITT AND K.A. ROSS, *Abstract harmonic analysis, VOL.1. (VOL.2.)* Springer-Verlag, Berlin, 1979.(1970).

[**Re**] H. REITER, *Classical harmonic analysis and locally compact groups*. Oxford Mathematical Monographs. 1968.

[**Vo**] M. VOGEL, Spectral synthesis on algebras of orthogonal polynomial series, *Math. Zeit.* (1987), 99-116.