LIBERTAS MATHEMATICA, vol XXXI (2011)

Euclidean Semimodules[1]

E. MEHDI-NEZHAD and A. M. RAHIMI

Abstract. In this article we introduce the notion of the Euclidean semimodules over Euclidean semirings. A Euclidean semimodule over a semiring R is a natural extension of a Euclidean semiring. Any subtractive subsemimodule (= k-subsemimodule) of a Euclidean R-semimodule is a cyclic R-semimodule. We show that every Euclidean semimodule over a semiring has always a universal side divisor and consequently, a cyclic semimodule with no universal side divisors can never be Euclidean. It is shown that a multiplicatively cancellative cyclic R-semimodule is Euclidean if and only if R is a Euclidean semiring. Moreover, we also prove the main result that a commutative semiring R is Euclidean if and only if every cyclic semimodule over R is Euclidean if and only if its endomorphism semiring is a Euclidean semiring for all cyclic R-semimodules M. It is shown that the homomorphic image of a Euclidean semimodule is also Euclidean and every nonempty subset of a Euclidean semimodule has a greatest common divisor.

Keywords: Euclidean semimodule, Euclidean semiring, subtractive ideal (= k-ideal), subtractive subsemimodule (= k-subsemimodule), endomorphism semiring.

Mathematics Subject Classification (2000): 16Y60.

1 Introduction and Preliminaries

In this article we introduce the notion of the *Euclidean semimodules* over Euclidean semirings. The notion of a Euclidean module over a commutative ring as a natural extension of the Euclidean rings was studied in [4]. A Euclidean semimodule over a semiring R is a natural extension of a Euclidean semiring. In this Section, we recall and review some basic properties and definitions together with some examples and show that for every Euclidean semimodule, there always exists a submultiplicative Euclidean function. In Section 2, it is shown that the homomorphic image of a Euclidean semimodule is Euclidean and every subtractive subsemimodule of a Euclidean semimodule is cyclic. Also, we show that a cancellative cyclic semimodule A over a semiring R is a Euclidean R-semimodule if and only if R is a Euclidean semiring. Actually, in Section 4, we extend this result to a more general case by relaxing the cancellative condition from the hypothesis. In Section 3, we discuss the concept of the greatest common divisors in the semimodules and show that for

[1]This research was in part supported by grant No. 88130015 from IPM.

any Euclidean semimodule A in which every cyclic subsemimodule is subtractive, then every nonempty finite subset of A has a greatest common divisor. In Section 4, we study the endomorphism semiring of a Euclidean semimodule and show that it is commutative. Also, we show that every semiring R is Euclidean if and only if every cyclic R-semimodule is Euclidean if and only if its endomorphism semiring is Euclidean. Finally, in Section 5, we extend the notion of the universal side divisors of commutative rings to semimodules and show that every Euclidean semimodule contains a universal side divisor which consequently implies that a cyclic semimodule with no universal side divisors is never Euclidean.

In this paper, all semirings (unless otherwise indicated) are commutative with identity $1 \neq 0$ and all semimodules are unitary. That is, a semimodule A over a semiring R is said to be unitary provided $1a = a$ for all a in A and the identity element 1 of R. Also, we assume for each a in an R-semimodule A and $0 \in R$, then $0a = 0_A$. A is said to be *multiplicatively cancellative* if For any r and s in R; and nonzero element x in A, $rx = sx$ implies $r = s$. If B is a nonempty subset of a semimodule A over a semiring R, then RB the subsemimodule generated by B is the set of all finite sums $\sum_{i=1}^{n} r_i b_i$, where $r_i \in R$ and $b_i \in B$. For any nonempty subset T of R, TA is defined to be the set of all finite sums $\sum_{i=1}^{n} t_i a_i$, where $t_i \in T$ and $a_i \in A$. A commutative semiring R is said to be a *semidomain* (= *entire*) if $ab = 0$ with $a, b \in R$, then either $a = 0$ or $b = 0$. A *semifield* is a commutative semiring in which the non-zero elements form a group under multiplication. A subset I of a semiring R will be called an ideal if $a, b \in I$ and $r \in R$ implies $a + b \in I$ and $ra \in I$. A *subtractive ideal* (= k-ideal) I is an ideal such that if $x, x + y \in I$, then $y \in I$ (so (0) is a subtractive ideal (= k-ideal) of R). Similarly, the definition of a *subtractive subsemimodule* (= k-subsemimodule) can be extended to semimodules as in the case of semirings. For a detailed study of semirings and semimodules, reader is referred to [3]. Also, for the study of subgroups of additive monoids, see [2].

For the sake of completeness, before stating the definition of a Euclidean R-semimodule, we write the definition of a (left) Euclidean semiring together with some examples (counterexample) exactly as given in Chapter 11 of [3] as follows.

Let N be the set of all nonnegative integers. A left Euclidean norm δ defined on a semiring R is a function $\delta : R\backslash\{0\} \to N$ satisfying the following condition:

($*$) If a and b are elements of R with $b \neq 0$, then there exist elements q and r of R satisfying $a = qb + r$ with $r = 0$ or $\delta(r) < \delta(b)$.

A right Euclidean norm is defined similarly, except that in condition ($*$) we have $a = bq + r$. A semiring R is left [resp. right] Euclidean if and only if there exists a left [resp. right] Euclidean norm defined on R. For commutative semirings, needless to say, the notions of left and right Euclidean norm coincide.

Example 1.1. The semiring N of nonnegative integers is Euclidean if we define the Euclidean norm δ by $\delta : n \mapsto n$ or $\delta : n \mapsto n^2$.

Example 1.2. Let $S[t]$ be the semiring of polynomials in the indeterminate t over a division semiring S and let $\equiv$ be the congruence relation on $S[t]$ defined by sum $a_i t^i \equiv \sum b_i t^i$ if and only if $a_1 t + a_0 = b_1 t + b_0$. Let R be the factor semiring $S[t]/\equiv$. Then there exists a left Euclidean norm $\delta : R\backslash\{0\} \to N$ defined by setting $\delta(\sum a_i t^i / \equiv) = 1$ if $a_1 \neq 0$ and equals to 0 if $a_1 = 0$ but $a_0 \neq 0$.

Example 1.3. Let R be the subsemiring of Q^+ (the set of nonnegative rationals) defined by $R = \{q \in Q^+ | q = 0 \text{ or } q \geq 1\}$ and suppose that we have a left Euclidean norm $\delta : R\backslash\{0\} \to N$. Let $0 < a < b$ be elements of R. If $\delta(a) \geq \delta(b)$, then there would have to exist elements q and r of R satisfying $a = qb + r$, where $r = 0$ or $\delta(r) < \delta(b)$. But $a < b$ implies that $a < qb$ for all $0 \neq q \in R$ and $a = 0b + r$ leads to the contradiction $\delta(a) = \delta(r) < \delta(b)$. Thus, $a < b$ implies that $\delta(a) < \delta(b)$ for all $0 \neq a, b \in R$. Hence, $R\backslash\{0\}$ is order-isomorphic to the subset $im(\delta)$ of N, which is impossible. Thus, no left Euclidean norm can be defined on R, and so R is not a left Euclidean semiring.

Definition 1.4. Let N be the set of nonnegative integers and A a unitary R-semimodule over the semiring R. A is a Euclidean R-semimodule if there is a function $\phi : A\backslash\{0\} \to N$ such that if a and b are elements of A with $b \neq 0$, then there exist $r \in R$ and $c \in A$ such that $a = rb + c$ with $c = 0$ or $c \neq 0$ and $\phi(c) < \phi(b)$.

Example 1.5. Every Euclidean semiring R is a Euclidean R-semimodule.

Example 1.6. From the above definition, we can easily verify that every *simple semimodule* over a semiring is a Euclidean semimodule. A semimodule is simple if the only subsemimodules it contains are (0) and itself.

Example 1.7. Clearly, if A is a Euclidean semimodule over a semiring R, then any nonzero subtractive subsemimodule of A is a Euclidean R-semimodule. Thus, every nonzero subtractive ideal of a Euclidean semiring R is a Euclidean R-semimodule.

Example 1.8. Let $R[X]$ be the Euclidean ring of polynomials over the field of real numbers with the Euclidean function $\phi(f) = \deg(f)$ for each nonzero element $f \in R[X]$. It is clear that the ideal $I = (X^2)$ is a Euclidean $R[X]$-module which is not a Euclidean ring under the function ϕ since $X^3 = XX^2 + 0$ and $X \notin I$.

Theorem 1.9. *Let A be an R-semimodule over a semiring R. If ϕ is a Euclidean function defined on A, then there exists another Euclidean function ϕ^* defined on A and satisfying:*

(1) $\phi^*(a) \leq \phi(a)$ *for all a in $A\backslash\{0\}$; and*

(2) $\phi^*(b) \leq \phi^*(rb)$ *for all b in A, and r in R satisfying $rb \neq 0$.*

Proof. For each $0 \neq a \in R$, set $\phi^*(a) = \min\{\phi(ra) | ra \neq 0\}$. The function ϕ^* clearly satisfies (1) and (2), so all we have to show is that it is indeed a Euclidean function on A. Let a and b be nonzero elements of A satisfying $\phi^*(a) \geq \phi^*(b)$. Then there exists an element s of R such that $\phi^*(sb) = \phi(sb)$. Then $\phi(a) \geq \phi(sb)$ and so there exist elements q in R and c of A such that $a = qsb + c$, where $c = 0$ or $\phi(c) < \phi(sb)$. In the second case, we have $\phi^*(c) \leq \phi(c) < \phi(sb) = \phi^*(b)$. Thus, ϕ^* is a Euclidean function on A. $\square$

Thus, by virtue of the above result, if (A, ϕ) is a Euclidean R-semimodule over a semiring R, we can, without loss of generality, assume that ϕ satisfies the condition that $\phi(b) \leq \phi(rb)$ for all $0 \neq b \in A$ and all r in R such that $rb \neq 0$. A Euclidean function satisfying this condition is said to be *submultiplicative*.

2 Some Basic Properties

In this section, we study some basic algebraic properties of the Euclidean semimodules whereas the counterparts of some of them on the Euclidean semirings are given in Chapter 11 of [3].

Theorem 2.1. *Every subtractive subsemimodule of a Euclidean semimodule over a semiring R is a cyclic R-semimodule. Consequently, every Euclidean semimodule is cyclic since every R-semimodule is automatically a subtractive subsemimodule of itself.*

Proof. If B is a nonzero subtractive subsemimodule of a Euclidean R-semimodule A with Euclidean function ϕ, choose an element $b \in B$ such that $\phi(b)$ be the least integer in the set of nonnegative integers $\{\phi(x)|x \neq 0, x \in B\}$. If $a \in B$, then for some $r \in R$ and $c \in A$, $a = rb + c$ with $c = 0$; or $c \neq 0$ and $\phi(c) < \phi(b)$. Now, the minimality of $\phi(b)$ and the fact that $c \in B$ since B is a subtractive subsemimodule of A will imply the desired result. Note that since A is automatically a subtractive subsemimodule of itself, thus A is a cyclic semimodule over R. □

From the above theorem, we know that every Euclidean semimodule is a cyclic semimodule. So, from the fact that every simple semimodule is a Euclidean semimodule, we have the following containment relations:

$$\{\text{simple semimodules}\} \subseteq \{\text{Euclidean semimodules}\} \subseteq \{\text{cyclic semimodules}\}.$$

But a Euclidean semimodule should not be a simple semimodule and there exists a cyclic semimodule which is not Euclidean. We give the following examples.

Example 2.2. Consider the Z-module Z. Obviously, it is a Euclidean module since Z is a Euclidean domain. But Z is not a simple module since it has a proper submodule $2Z$.

Example 2.3. Consider the N-semimodule N of nonnegative integers. Obviously, it is a Euclidean semimodule since N is a Euclidean semimodule. But N is not a simple semimodule since it has a proper subsemimodule $2N$.

Example 2.4. Consider the integral domain $Z[\theta] = \{a + b\theta | a, b \in Z\}$, where $\theta = (1 + (\sqrt{-19}))/2$. In [1], Campoli proved that $Z[\theta]$ is a principal ideal domain which is not a Euclidean domain. Hence, $Z[\theta]$-module $Z[\theta]$ is a cyclic module but it is not a Euclidean module.

Theorem 2.5. *The homomorphic image (surjective homomorphism) of a Euclidean semimodule is also a Euclidean semimodule.*

Proof. Let A be a Euclidean semimodule over a semiring R with Euclidean function ϕ. Suppose $f : A \to B$ is a surjection of R-semimodules of A and B. We define $\phi_B : B\backslash\{0\} \to N$ as follows: $b \mapsto \min\{\phi(a)|a \in f^{-1}(b)\}$. Note that for any $0 \neq b \in B$, $f^{-1}(b)$ does not contain 0_A since $f(0_a) = f(0_R \dot{0}_A) = 0_R f(0_A) = 0_B$ by the definition of a semimodule. Clearly, ϕ_B is well defined since f is a surjection. For all b_1 in B and b_2 ($\neq 0$) in B, we can choose $a_1, a_2 \in A$ such that $f(a_1) = b_1$, $f(a_2) = b_2$ and $\phi_B(b_2) = \phi(a_2)$. Obviously, $a_2 \neq 0$ since $b_2 \neq 0$. Since A is a Euclidean semimodule, there exists an element r in R

and a_3 in A such that $a_1 = ra_2 + a_3$, where $a_3 = 0$; or $a_3 \neq 0$ and $\phi(a_3) < \phi(a_2)$. So $f(a_1) = rf(a_2) + f(a_3)$, that is, $b_1 = rb_2 + b_3$ (here we denote $f(a_3) = b_3$). If $b_3 \neq 0$, then $(\phi_B(b_3) = \min\{\phi(a)|f(a) = b_3\} \leq \phi(a_3) < \phi(a_2) = \phi_B(b_2)$, that is, $\phi_B(b_3) < \phi_B(b_2)$. Hence, ϕ_B is the desired Euclidean function for B and B is a Euclidean R-semimodule. □

But the converse of the above result is not true in general. For example, consider the Z-module short exact sequence

$$0 \to Z \xrightarrow{\iota} Z \oplus Z_2 \xrightarrow{\pi} Z_2 \to 0.$$

Obviously, Z is a Euclidean Z-module since it is a Euclidean domain. Also, Z_2 is Euclidean as a homomorphic image of Z. But $Z \oplus Z_2$ is not Euclidean since it is not cyclic. From this example, we can conclude that the class of Euclidean semimodules under the action of direct sum is not closed.

Remark 2.6. By using submultiplicative property of a Euclidean function, it is not difficult to show that for any $c \in A$ and any generator a of A, we have always $\phi(a) \leq \phi(c)$. Moreover, for any generator a of a Euclidean semimodule A over a semiring R with Euclidean function ϕ, $\phi(b) = \phi(a)$ if and only if b is a generator of A. Also, for a Euclidean semiring R with Euclidean norm (function) δ, in Proposition 11.11 [3], it is shown that $\delta(1_R) \leq \delta(r)$ for each $r \in R$. Moreover, it is shown that $\delta(r) = \delta(1_R)$ if and only if r is a unit in R. Recall that for any nonzero element r in a Euclidean ring R with Euclidean function ϕ, $\phi(1_R) \leq \phi(r)$, and $\phi(r) = \phi(1_R)$ if and only if r is a unit in R.

Theorem 2.7. *A cancellative cyclic semimodule A over a semiring R is a Euclidean R-semimodule if and only if R is a Euclidean semiring.*

Proof. For the sufficiency, let R be a Euclidean semiring with Euclidean function ϕ. Let $A = Rx$ be a cancellative cyclic semimodule over R. Define $\psi : A\backslash\{0\} \to N$ as $\psi(a) = \psi(rx) = \phi(r)$ for each nonzero a in A, where $a = rx$ for some r in R. The proof can be followed directly from the definition if we show that ψ is a well-defined function. Suppose, y is a generator of A. Thus, $x = ry = rsx$, for some $r, s \in R$, implies $1_R = rs$ which makes $\psi(y) = \psi(sx) = \phi(s) = \phi(1_R) = \psi(x)$. Now, suppose $a = rx = sy$ for some nonzero r and s in R and y a generator of A. Hence, $a = rx = sy = stx$, for some $t \in R$, implies $r = st$ which makes $\psi(a) = \psi(rx) = \phi(r) \geq \phi(s) = \psi(sy)$. Again, by a similar argument, $\psi(sy) = \phi(s) \geq \phi(r) = \psi(rx)$. For the necessary part, suppose $A = Rx$ and define $\psi : R\backslash\{0\} \to N$ as $\psi(r) = \phi(rx)$ for each nonzero r in R. Next, we show that ψ is a well-defined function on $R\backslash\{0\}$ and the rest of the proof which can be followed directly from the definition is left to the reader. Suppose, $\psi(r) = \phi(rx)$ and $\psi(r) = \phi(ry)$ for an arbitrary generator y of A. Since A is a Euclidean R-semimodule, $\phi(rx) = \phi(rsy) = \phi(sry) \geq \phi(ry)$ and similarly $\phi(ry) = \phi(rtx) = \phi(trx) \geq \phi(rx)$. Consequently, $\phi(rx) = \phi(ry)$. □

Remark 2.8. It is not difficult to show that in a nonzero cancellative cyclic R-semimodule $A = Rx$, rx is a generator of A if and only if r is a unit in R. Note that for the sufficient part, A need not be a cancellative R-semimodule.

3 The Greatest Common Divisors in the Euclidean Semimodules

As a generalization of the divisors in a semiring, we define divisors in the semimodules. Let A be a semimodule over a semiring R. We say an element b of A is a *divisor* of a in A whenever $a = rb$ for some r in R. If a is an element of a semimodule A over a semiring R, then we denote by $D(a)$ the set of all divisors of a in A. That is to say, $D(a) = \{b \in A | a \in Rb\} = \{b \in A | Ra \subseteq Rb\}$. Since b in $D(b)$ for all b in A, it is clearly true that b in $D(a)$ if and only if $D(b) \subseteq D(a)$.

If B is a nonempty subset of a semimodule A over a semiring R, then the set of common divisors of B is $CD(B) = \cap\{D(a) | a \in B\} = \{b \in A | RB \subseteq Rb\}$. An element b in $CD(B)$ is a *greatest common divisor* of B if and only if $CD(B) = D(b)$.

Theorem 3.1. *If B is a nonempty subset of a semimodule A over a semiring R, then an element b of A is a greatest common divisor of B if and only if the following conditions are satisfied:*

(1) *$RB \subseteq Rb$;*

(2) *if $c \in A$ satisfies $RB \subseteq Rc$, then $Rb \subseteq Rc$.*

Proof. Assume that b is a greatest common divisor of B. Then $b \in CD(B)$ and so $b \in D(a)$ for each a in B. Thus, $Ra \subseteq R$b for each a in B, implying that $RB \subseteq Rb$. Moreover, if $RB \subseteq Rc$ for some element c of A, then $c \in CD(B) = D(b)$ and so $Rb \subseteq Rc$. Conversely, assume conditions (1) and (2) are satisfied. By (1), $b \in CD(B)$ and so $D(b) \subseteq CD(B)$. By (2), if $c \in CD(B)$, then $RB \subseteq Rc$ and so $Rb \subseteq Rc$. Hence, $c \in D(b)$, proving that $CD(B) \subseteq D(b)$ and thus yielding equality. □

Corollary 3.2. *If every subtractive subsemimodule of a semimodule A over a semiring R is cyclic, then every nonempty subset of A has a greatest common divisor.*

Proof. Let B be a nonempty subset of A. Then $RB = R$ or RB is a subsemimodule of A. Hence, by hypothesis, there exists an element b of A satisfying $RB = Rb$. By Theorem 3.1, b is a greatest common divisor of B. □

Theorem 3.3. *Let a, b, and c be elements of a semimodule A over a semiring R. If d is a greatest common divisor of a, b and e is a greatest common divisor of c, d, then e is a greatest common divisor of a, b, c.*

Proof. By definition, $D(e) = D(d) \cap D(c) = D(a) \cap D(b) \cap D(c) = CD(a, b, c)$. □

If a and b are elements of a semimodule A over a semiring R, then clearly $CD(a, b) \subseteq CD(a + b, b)$. We now investigate the conditions for having equality.

Theorem 3.4. *The following conditions on a semimodule A over a semiring R are equivalent:*

(1) *$CD(a, b) = CD(a + b, b)$ for all $a, b \in A$;*

(2) *every cyclic subsemimodule of A is subtractive.*

Proof. Assume (1) holds and let Rd be a cyclic subsemimodule of A. If a and $a+b$ belong to Rd, then $d \in CD(a+b,a) = CD(a,b)$ and so $b \in Rd$. Therefore, Rd is a subtractive subsemimodule. Conversely, assume (2) is true and let $a, b \in A$. If $d \in CD(a+b,b)$, then $a+b$ and b both belong to Rd and so, by (2), $a \in Rd$. Therefore, $d \in CD(a,b)$. □

Theorem 3.5. *The following conditions on a Euclidean semimodule A over a semiring R are equivalent:*

(1) *every subsemimodule of A is cyclic and subtractive;*

(2) *there exists a Euclidean function ϕ defined on A satisfying the condition that if $a = qb + c$ for some c in $A\backslash\{0\}$ and $\phi(c) < \phi(b)$, then $a \notin Rb$.*

Proof. (1) ⇒ (2): By Theorem 1.9, we know that there exists a Euclidean function ϕ on A satisfying the condition that $\phi(a) \leq \phi(ra)$ for all r in R and a in $A\backslash\{0\}$. Assume that $a = qb + c$ for some c in $A\backslash\{0\}$ and $\phi(c) < \phi(b)$. If $a \in Rb$, then by (1), we must have $c = rb$ for some r in R and so $\phi(c) \geq \phi(b)$, which is a contradiction. Thus, $a \notin Rb$.

(2) ⇒ (1): Assume that $a, b \in A$ and that $t \in CD(a+b,b)$. Then we can write $a+b = rt$ and $b = st$ for elements r and s of R. By the choice of ϕ, we know that $\phi(a) \geq \phi(t)$ and so either $a = qtora = qt + c$ for some $0 \neq c \in A$ satisfying $\phi(c) < \phi(t)$. But in the latter case, we have $rt = (s+q)t + c$, which again contradicts the stated condition. Thus, we must have $a = qt$ and so $t \in D(a)$. Since $t \in D(b)$ by the choice of t, we have $t \in CD(a,b)$. Thus, A is a cyclic semimodule in which every cyclic subsemimodule is subtractive by Theorem 3.4. □

Theorem 3.6. *If A is a Euclidean semimodule over a semiring R in which every cyclic subsemimodule is subtractive, then any nonempty finite subset B of A has a greatest common divisor.*

Proof. By Theorem 3.3, it suffices to consider the case of $B = \{a,b\}$. If $a = b = 0$, then 0 is a greatest common divisor of $\{a,b\}$ and we are done. Hence, without loss of generality, we can assume that $b \neq 0$. Since every cyclic subsemimodule of A is subtractive, we know by Theorem 3.5 that there exists a Euclidean function ϕ defined on A satisfying the condition that if $a = qb + c$ for some c in $A\backslash\{0\}$ satisfying $\phi(c) < \phi(b)$, then $a \notin Rb$. By repeated applications of ϕ, we can find elements $q_1, \ldots, q_{n+1}$ of R and $c_1, \ldots, c_n$ of $A\backslash\{0\}$ such that $a = q_1b + c_1$, $b = q_2c_1 + c_2$, $\ldots$, $c_{n-2} = q_nc_{n-1} + c_n$, $c_{n-1} = q_{n+1}c_n$ and $\phi(b) > \phi(c_1) > \cdots > \phi(c_n)$. (The process of selecting the q_i's and c_i's must indeed terminate after finitely-many steps since there are no infinite decreasing sequences of elements of N.) Working backwards, we then see that $c_{n-2} = [q_nq_{n+1} + 1]c_n$, $c_{n-3} = [q_{n-1}q_nq_{n+1} + q_{n-1} + q_n + 1]c_n$, etc. until we establish that $c_n \in CD(\{a,b\})$. Conversely, assume that $d \in CD(\{a,b\})$. By Theorem 3.5,, we see that $d \in D(c_1), d \in D(c_2), \cdots, d \in D(c_n)$ and so $D(c_n) = CD(\{a,b\})$. Thus, c_n is a greatest common divisor of $\{a,b\}$. □

Recall that a module M over a ring is said to be a *uniform module* if the intersection of two nonzero submodules of M is again nonzero. We now extend this definition to the *uniform semimodules* over a semiring. Since the intersection of subtractive subsemimodules of a semimodule over a semiring is subtractive, we say a semimodule A is said to be uniform whenever the intersection of any two nonzero subtractive subsemimodules of A is a nonzero subsemimodule of A. We next state the following theorem For a *torsion-free* Euclidean R-semimodule.

Theorem 3.7. *Let A be a torsion-free Euclidean semimodule over a semiring R. Suppose each greatest common divisor of any pair of elements a and b of A is a linear combination of a and b. That is,* $\gcd(a, b) = r_1a + r_2b$ *for some r_1 and r_2 in R. Then A is a uniform semimodule.*

Proof. Suppose A_1 and A_2 are two nonzero subtractive subsemimodules of A, both of A_1 and A_2 are cyclic since A is Euclidean. We may assume that $A_1 = Ra_1$ $(a_1 \neq 0)$ and $A_2 = Ra_2$ $(a_2 \neq 0)$. By hypothesis, we know that there exist $r_1, r_2 \in R$ such that $\gcd(a_1, a_2) = r_1a_1 + r_2a_2$ (here we denote $a_0 = \gcd(a_1, a_2) = r_1a_1 + r_2a_2$). Because a_0 divides a_1, there exists $r_3 \in R$ such that $a_1 = r_3a_0$. Hence, $a_1 = r_3(r_1a_1 + r_2a_2)$ and $a_1 = r_3r_1a_1 + r_3r_2a_2$. Since A_1 is subtractive, then $r_3r_2a_2 \in A_1$. If $r_3r_2a_2 \neq 0$, then $r_3r_2a_2 \in Ra_1 \cap Ra_2$ and thus $Ra_1 \cap Ra_2 \neq 0$. If $r_3r_2a_2 = 0$. We know $r_3 \neq 0$ since $a_1 \neq 0$ and so $r_2a_2 = 0$ since A is torsion-free. Thus, $a_0 = r_1a_1$. While a_0 also divides a_2, there exists $r_4 \in R$ such that $a_2 = r_4a_0 = r_4r_1a_1$. Thus, $Ra_1 \cap Ra_2 \neq 0$ since $a_2 \in Ra_1 \cap Ra_2$. Hence, A is a uniform semimodule. □

In the above theorem, the condition that A is "torsion-free" can not be removed. Otherwise, the theorem may be false.

Example 3.8. Consider the Z-module Z_6. Obviously, Z_6 is Euclidean since it is a homomorphic image of Z. But it is not a uniform module since $A_1 = \{\overline{0}, \overline{2}, \overline{4}\}$ and $A_2 = \{\overline{0}, \overline{3}\}$ are two nonzero submodules such that $A_1 \cap A_2 = \overline{0}$.

4 The Endomorphism Semiring of a Euclidean Semimodule

In this paper, we will write endomorphisms of left semimodules on the right side of the elements of the semimodule and endomorphisms of right semimodules on the left. Under this notation, $(a)(fg) = g(f(a))$, where $f, g \in \mathrm{End}({}_RA)$. We define similar to the definition of a *balanced bimodule* that a bisemimodule ${}_RA_S$ is a balanced bisemimodule if "left and right multiplications" λ and ρ are both surjective, where $\lambda : R \to \mathrm{End}(A_S)$ and $\rho : S \to \mathrm{End}({}_RA)$ such that for r in R, x in A and s in S, $\lambda(r) : x \mapsto rx$ and $\rho(s) : x \mapsto xs$. To prove the main result of this section, we need the following lemma.

Lemma 4.1. *Let R be a commutative semiring and A a cyclic R-semimodule. Then*

(1) ${}_RA_R$ *is a balanced bisemimodule;*

(2) $\mathrm{End}({}_RA)$ *is a commutative semiring.*

Proof. (1) Suppose $A = Ra_0 = a_0R$, where $a_0 \in A$. Given $\lambda : R \to \mathrm{End}(A_R)$ and $\rho : R \to \mathrm{End}({}_RA)$ For any $f \in \mathrm{End}(A_R)$, there exists an $r \in R$ such that $f(a_0) = a_0r$. Then for any $a_0r_1 \in A$, $f(a_0r_1) = f(a_0)r_1 = (a_0r)r_1 = (ra_0)r_1 = r(a_0r_1)$. Hence, there exists an r in R such that $\lambda(r) = f$, that is, λ is an epimorphism. Similarly, ρ is also an epimorphism. Thus, ${}_RM_R$ is a balanced bisemimodule.

(2) Suppose $f_1, f_2 \in \mathrm{End}({}_RA)$, then there exist $r_1, r_2 \in R$ such that $\rho(r_1) = f_1$ and $\rho(r_2) = f_2$ since ${}_RA_R$ is a balanced bisemimodule by (1). For all a in A, $(a)f_1f_2 = (ar_1)f_2 = (ar_1)r_2 = (ar_2)r_1 = (a)f_2f_1$. Hence, $f_1f_2 = f_2f_1$ and $\mathrm{End}({}_RA)$ is commutative. $\square$

We now can write the following corollary since every Euclidean semimodule is cyclic.

Corollary 4.2. *Suppose R is a commutative semiring and A is a Euclidean semimodule over R. Then ${}_RA_R$ is a balanced bisemimodule and* $\mathrm{End}({}_RA)$ *is a commutative semiring.*

We now can prove the main result of this section as follows.

Theorem 4.3. *For a commutative semiring R, the following statements are equivalent:*

(1) *R is a Euclidean semiring;*

(2) *every cyclic R-semimodule is Euclidean;*

(3) *for every cyclic R-semimodule A,* $\mathrm{End}({}_RA)$ *is a Euclidean semiring;*

(4) *for every cyclic R-semimodule A, every cyclic* $\mathrm{End}({}_RA)$*-semimodule is Euclidean.*

Proof. (1) $\Rightarrow$ (2): If R is a Euclidean semiring, then R as an R-semimodule over itself is a Euclidean semimodule. Since (similar to the case of cyclic modules) every cyclic R-semimodule $A = Rx$ is a homomorphic image of R, given by $r \mapsto rx$, then by applying Theorem 2.5 that the homomorphic image of a Euclidean semimodule is Euclidean, we can conclude that every cyclic R-semimodule is Euclidean.

(2) $\Rightarrow$ (1): It is obvious since R is a cyclic R-semimodule over itself.

(2) $\Rightarrow$ (3): We prove that $\mathrm{End}({}_RA)$ is a Euclidean semiring. Let A be a cyclic semimodule, then by assumption A is a Euclidean semimodule. Suppose $A = Ra_0 = a_0R$ ($a_0 \neq 0$) and ϕ is its associated Euclidean function. If $0 \neq f \in \mathrm{End}({}_RA)$, then $f(a_0) \neq 0$. For if $f(a_0) = 0$, then $f(Ra_0) = f(A) = 0$ and $f = 0$ which is a contradiction. So we may define $\Phi : \mathrm{End}({}_RA)\backslash\{0\} \to N$ as follows: $f \mapsto \phi(f(a_0))$. For all $f_1 \in \mathrm{End}({}_RA)$ and $0 \neq f_2 \in \mathrm{End}({}_RA)$, then $f_2(a_0) \neq 0$. Since A is a Euclidean semimodule, there exist $r \in R$ and $a_3 \in A$ such that $f_1(a_0) = rf_2(a_0) + a_3$, where $a_3 = 0$ or $\phi(a_3) < \phi(f_2(a_0))$. Suppose $a_3 = r'a_0 = a_0r'$. We now define $f_3 : A \to A$ as $a \mapsto ar$, and $f_4 : A \to A$ as $a \mapsto ar'$. We can easily verify that $f_3, f_4 \in \mathrm{End}({}_RA)$. Suppose $f_2(a_0) = r_1a_0$, then $(a_0)f_2f_3 = (r_1a_0)f_3 = r_1(a_0f_3) = r_1(a_0r) = (r_1a_0)r = rf_2(a_0)$. So $(a_0)f_1 = (a_0)f_2f_3 + (a_0)f_4 = (a_0)(f_2f_3 + f_4)$. Since $A = Ra_0 = a_0R$, then $f_1 = f_2f_3 + f_4$. If $a_3 = 0$, then $f_4 = 0$. If $a_3 \neq 0$, then $f_4 \neq 0$ and $\Phi(f_4) = \phi(f_4(a_0)) = \phi(a_3) < \phi(f_2(a_0)) = \Phi(f_2)$. Hence, Φ is the right Euclidean function (norm) for $\mathrm{End}({}_RA)$ and thus $\mathrm{End}({}_RA)$ is a right Euclidean semiring. But $\mathrm{End}({}_RA)$ is commutative by Lemma 4.1, so $\mathrm{End}({}_RA)$ is a Euclidean semiring.

(3) $\Rightarrow$ (2): Let A be a cyclic R-semimodule and $\mathrm{End}({}_RA)$ a Euclidean semiring. Suppose Φ is the associated Euclidean function of $\mathrm{End}({}_RA)$ and $A = Ra_0 = a_0R$, where $a_0 \in A$.

If $0 \neq a \in A$, let $a = ra_0 = a_0 r$. We can define an $f : A \to A$ as $a \mapsto ar$ such that $f(a_0) = a_0 r = a$. So for all $0 \neq a \in A$, the set $\{\Phi(f) | (f(a_0) = a\}$ is a nonempty subset of N. Now we can define $\phi : A \backslash \{0\} \to N$ as follows: $a \mapsto \min\{\Phi(f) | f(a_0) = a\}$. Obviously, this function is well defined. We shall prove that ϕ is the appropriate Euclidean function for A. For all $a_1 \in A$ and $0 \neq a_2 \in A$, we can choose an f_2 in $\mathrm{End}(_R A)$ such that $f_2(a_0) = a_2$ and $\phi(a_2) = \Phi(f_2)$. Obviously, $f_2 \neq 0$. If $a_1 = 0$, then $0 = 0a_2 + 0$. If $a_1 \neq 0$, then we can choose an f_1 in $\mathrm{End}(_R A)$ such that $f_1(a_0) = a_1$. Since $\mathrm{End}(_R A)$ is a Euclidean semiring, then there exist f_3, and f_4 in $\mathrm{End}(_R A)$ such that $f_1 = f_2 f_3 + f_4$, where $f_4 = 0$ or $\Phi(f_4) < \Phi(f_2)$. So $(a_0)f_1 = (a_0)(f_2 f_3 + f_4) = (a_0)f_2 f_3 + (a_0)f_4 = (a_2)f_3 + a_3$ (here, we denote $f_4(a_0) = a_3$). That is, $a_1 = f_3(a_2) + a_3$. According to Lemma 4.1, we know that $_R A_R$ is a balanced bisemimodule, thus there exists r in R such that $f_3(a_2) = a_2 r = ra_2$. So $a_1 = ra_2 + a_3$. If $f_4 = 0$, then $a_3 = f_4(a_0) = 0$. If $f_4 \neq 0$, then $\phi(a_3) = \min\{\Phi(f) | f(a_0) = a_3\} \leq \Phi(f_4) < \Phi(f_2) = \phi(a_2)$. That is, $\phi(a_3) < \phi(a_2)$. Hence, ϕ is the appropriate Euclidean function for A and consequently, A is a Euclidean semimodule.

(3) $\Leftrightarrow$ (4): It is a special case of (1) $\Leftrightarrow$ (2) since $\mathrm{End}(_R A)$ is commutative for any cyclic R-semimodule A. □

From the above theorem, we can conclude the following corollary.

Corollary 4.4. *Suppose R is a commutative semiring and A is a Euclidean semimodule over R, then* $\mathrm{End}(_R A)$ *is a Euclidean semiring.*

5 The Universal Side Divisors of Cyclic Semimodules

In [6], it is shown that an integral domain with no *universal side divisors* can not be Euclidean. Also, in [4], the concept of the *side divisors* from commutative rings is extended to the Euclidean modules over a commutative ring R and shown that a torsion-free cyclic module with no universal side divisors is not Euclidean. In this section, we extend the notion of the universal side divisors from an integral domain to a semiring and semimodule over a commutative semiring. We show that every Euclidean semimodule has a universal side divisor. For a detailed study of the "side divisors" and the "universal side divisors" in a commutative ring, reader is referred to [5].

Definition 5.1. Suppose A is a cyclic semimodule over a semiring R. Let $G_0(A)$ (resp., $U_0(R)$) be the set of all generators (resp., units) of A (resp., R) together with zero. Also, suppose that $A \backslash G_0(A)$ (resp., $R \backslash U_0(R)$) is not an empty set. An element a in $A \backslash G_0(A)$ (resp., $R \backslash U_0(R)$) is said to be a side divisor of an element b in A (resp., R) whenever $b = qa + c$ for some q in R and c in $G_0(A)$ (resp., $U_0(R)$). We shall call an element $u \in A \backslash G_0(A)$ (resp., $R \backslash U_0(R)$) a universal side divisor in A (resp., R) whenever u is a side divisor of each element $x \in A$ (resp., R). That is, a universal side divisor u of A (resp., R) is an element of $A \backslash G_0(A)$ (resp., $R \backslash U_0(R)$) such that for every x in A (resp., R), then $x = qu + c$ for some elements q in R and c in $G_0(A)$ (resp., $U_0(R)$).

Theorem 5.2. *Every Euclidean semimodule A (resp., semiring R) over the semiring R has a universal side divisor provided that $A \backslash G_0(A)$ (resp., $R \backslash U_0(R)$) is nonempty.*

Proof. Let A be a Euclidean semimodule over a semiring R with Euclidean function ϕ. Let u be in $A\backslash G_0(A)$ with minimal value. Then for any x in A, there exist q in R and c in A such that $x = qu + c$, where $c = 0$ or $\phi(c) < \phi(u)$. Hence, minimality of $\phi(u)$ implies that c is an element of $G_0(A)$. Thus, u is a universal side divisor in A by definition. The result for the semiring as a special case is clearly true since every semiring is a semimodule over itself. □

Corollary 5.3. *A cyclic semimodule A (over a semiring) with $A \neq G_0(A)$ (resp., semiring R with $R \neq U_0(R)$) is not Euclidean provided that it has no universal side divisors.*

Proof. See the above theorem. □

Remark 5.4. It is not difficult to show that in a multiplicatively cancellative cyclic R-semimodule $A = Rx$ over a semiring R, rx is a generator of A if and only if r is a unit in R. Note that for the sufficient part, A need not be a multiplicatively cancellative R-semimodule. Thus, $a = rx$ is in $A\backslash G_0(A)$ if and only if r is in $R\backslash U_0(R)$, or equivalently, $a = rx \in G_0(A)$ if and only if $r \in U_0(R)$, where $U_0(R)$ is the set of all units in R together with zero. Consequently, by virtue of the next theorem, $A = Rx$ has no universal side divisors if and only if R has no universal side divisors.

Theorem 5.5. *For a cyclic semimodule $A = Rx$ over a semiring R, if r is a side divisor of s in R, then rx is a side divisor of sx in A. Conversely, for a multiplicatively cancellative cyclic R-semimodule $A = Rx$ if $a = rx$ is a side divisor of $b = sx$ in A, then r is a side divisor of s in R.*

Proof. The proof follows directly from the definition and the first part of the above remark. □

Corollary 5.6. *A multiplicatively cancellative cyclic semimodule A over a semiring R has no universal side divisors if and only if R contains no universal side divisors.*

References

[1] O. Campoli, *A principal ideal domain that is not a Euclidean domain*, American Mathematical Monthly, 95 (1988) 868-871.

[2] R. W. Gilmer, *Commutative semigroup rings*, the University of Chicago Press, Chicago, Illinois 1984.

[3] J. S. Golan, *The theory of semirings (with applications in mathematics and theoretical computer science)*, Pitman Monographs & Surveys in Pure and Appl. Math., vol. 54, Longman Scientific & Tech., Essex, 1992.

[4] A. M. Rahimi, *Euclidean modules*, Libertas Math., 22 (2002) 123-126.

[5] A. M. Rahimi, *Stable rings and side divisors*, Missouri Journal of Mathematical Sciences, Volume 10, No. 2, Spring (1998) 112-116.

[6] K. S. Williams, *Note on non-Euclidean principal ideal domains*, Math. Mag., May-June, (1975) 176-177.