

DESCENT IN RATIONAL HOMOTOPY THEORY

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Introduction

Although the phenomenon of descent is quite well known in number theory or algebraic geometry, it is only recently that it has found a place in the realm of algebraic topology. If two non-isomorphic algebraic objects defined over a field become isomorphic under an extension of the field (and the concomitant extension of scalars), then this situation is said to be an example of descent. For example, it is a familiar fact that there exist non-equivalent quadratic forms defined on a rational vector space which become equivalent when considered on the real vector space extension. With the advent of rational and real homotopy theory in the 1970's, the question of descent assumed a topological aspect. Very early it was discovered that the quadratic form example could be used to construct two spaces with the property that their rational homotopy types are distinct, but their real homotopy types are the same. Also, it was shown that certain properties of spaces exhibit no descent whatsoever. In particular, a space is \mathbb{Q} -formal (i.e. its rational homotopy type is determined by its rational cohomology algebra) if and only if it is k -formal for an extension k (i.e. its k -homotopy type is determined by its k -algebra). See [S], [NM], or [HS] for example.

The question of descent for homotopy classes of maps was first brought to my attention by Steve Halperin. Specifically, do there exist two

rationally non-homotopic maps which become homotopic when considered over the reals? Although the answer is not known in general, it will be demonstrated here that this situation cannot occur if the maps are rational equivalences. Since the proof of this result depends on the algebraic group structure of the automorphisms of the minimal model, we first review the fundamentals of rational homotopy theory. To make this approach concrete, we then explicitly construct (via circle actions) the distinct rational homotopy types of two 12-manifolds which coalesce to a single real homotopy type.

In the following, all spaces are assumed to be rational and simply connected with finite rational betti numbers. Therefore, all cohomology, homology and homotopy shall be understood to have rational coefficients without explicit denotation. When considering objects over the real numbers, we shall write either the subscript $_{\mathbb{R}}$ or the tensor product $\otimes \mathbb{R}$.

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1. Minimal Model Theory

To any space X there is associated a "minimal" differential graded algebra (DGA) model $M(X)$ which describes the homotopy type of X . (Recall that all spaces are assumed to be rational!) The model $M(X)$ has a very special structure which makes it amenable to both calculation and manipulation. As a graded algebra, $M(X) \cong L(V)$, where $L(V)$ denotes the commutative graded algebra freely generated by the graded vector space V . That is, $L(V)$ is the tensor product of a polynomial algebra on even dimensional basis elements of V and an exterior algebra on odd dimensional basis elements of V . Furthermore, $M(X)$ is provided with a differential d , a graded derivation of degree 1 satisfying $d^2 = 0$ and $d(M^+) \subset M^+ \cdot M^+$.

This last property is called "decomposability of the differential" and is useful in inductive proofs and constructions.

We may consider various algebraic objects associated to $M(X)$ like its cohomology $H^*(M(X))$ or its space of indecomposables $QM(X) = M(X)^+/M(X)^+ \cdot M(X)^+$. One of the things we mean when we say that $M(X)$ describes the homotopy type of X is that there exist natural isomorphisms, $H^*(M(X)) \cong H^*(X)$, $Q^i M(X) \cong \text{Hom}(\pi_i X, \mathbb{Q})$. (Here, $\pi_i X$ denotes the i^{th} homotopy group of X .) In fact, the construction of a minimal model for a space X mimics the construction of the Postnikov tower for X , so it is not surprising that the algebraic entity $M(X)$ contains all homotopy information about X . Furthermore, by reversing the process, for any given minimal DGA M we may construct a space which reflects the appropriate "homotopical" structure of M . Indeed, the set of (rational) homotopy types is in one-to-one correspondence with the isomorphism classes of minimal DGA's.

If $f: X \rightarrow Y$ is a map of spaces, then there is an induced map $F: M(Y) \rightarrow M(X)$. Indeed, there is a DGA homotopy theory which provides a correspondence between homotopy of spaces and homotopy of differential graded algebras. We shall not present this here because we make no use of the explicit definition. For all of the material so far the reader is referred to [GM] for example.

It is also possible to treat fibrations from the minimal model point of view (see [H1] for example). A fibration $F \rightarrow E \rightarrow B$ is modeled by an extension of DGA's

$$(L(V), d_B) \rightarrow (L(V \oplus W), D) \rightarrow (L(W), d_F)$$

where $M(B) \cong (L(V), d_B)$, $M(F) \cong (L(W), d_F)$ and $H^*(L(V \oplus W), D) \cong H^*(E)$.

Although the middle DGA will not be minimal in general, the "twisted" differential D is required to satisfy the following properties:

1. $D^2 = 0$.
2. $Dv = d_B v$.
3. $Dw = d_F w + \tau w$, where τ is a degree 1 derivation

$$M(F) \rightarrow M(F) \otimes M(B)^+, (M(B)^+ = \bigoplus_{i>0} M^i(B)).$$

In particular, the dual to the usual homotopy sequence of a fibration may be obtained as the long exact sequence associated to the extension's short exact sequence of cochain complexes of indecomposables.

To end this section on fundamentals we recall the work of Sullivan [S]. If M is a minimal DGA which is finitely generated as an algebra, then the group of DGA automorphisms $\text{Aut}(M)$ has the structure of a \mathbb{Q} -algebraic matrix group. The group of homotopy classes of automorphisms $h - \text{Aut}(M)$ has such a structure as well and the projection $\text{Aut}(M) \rightarrow h - \text{Aut}(M)$ is a morphism of algebraic groups. The kernel of this morphism is denoted by U and consists of all automorphisms which are homotopic to the identity. Sullivan showed that every element of U has the form $\exp(di + id)$, where $i \in \text{Der}^{-1}(M)$, the degree (-1) derivations of M (also see [H1]). In this way it can be seen that U is a unipotent algebraic group with nilpotent Lie algebra

$$L(U) = \{di + id \mid i \in \text{Der}^{-1}(M)\}$$

and that there are inverse bijections $L(U) \xrightleftharpoons[\log]{\exp} U$.

Remark

All that we have said for DGA's over \mathbb{Q} holds for DGA's over \mathbb{R} as well. To pass from \mathbb{Q} to \mathbb{R} we merely tensor with \mathbb{R} over \mathbb{Q} . In particular, if X is a space, then its real homotopy type is described by forming $M(X) \otimes \mathbb{R}$. Furthermore, if X is a smooth manifold, then its real homotopy type may be recovered directly from its DeRham algebra of smooth forms $\Omega(X)$.

2. An Example of Descent

In this section we will construct two distinct rational homotopy types (of 12-manifolds) which have the same real homotopy type. Before we can accomplish our construction, we must recall several basic results from the theory of transformation groups as well as the rational homotopy theory approach to compact group actions.

If a group G acts on a space X , then there is an associated bundle (called the Borel fibration),

$$X \rightarrow XG \rightarrow BG$$

where XG is the orbit space of $X \times EG$ under the "diagonal" action. Here, EG denotes the contractible free G -space and $BG = EG/G$. If X is a finite CW complex and G acts almost freely (i.e. each isotropy group is finite), then the Vietoris-Begle Theorem implies that,

$$XG \underset{\mathbb{Q}}{\sim} X/G \text{ and } \dim H^*(XG) < \infty.$$

In this sense, the total space of the Borel fibration is the homotopical version of the orbit space of a group action.

In the following we shall restrict to the situation $G = S^1$, the circle group. The classifying space is given by $BS^1 = \mathbb{CP}(\infty)$, infinite complex projective space. It is well known that S^1 acts almost freely on a space X if and only if the fixed set $F(S^1, X)$ is empty. Therefore, by using the following condition it becomes quite easy to check whether an action is almost free.

Fixed Point Condition (see [Hsi]). Suppose S^1 acts on a finite CW complex. Then $F(S^1, X) \neq \emptyset$ if and only if $H^*(BS^1) \rightarrow H^*(XS^1)$ is injective.

Recall that $H^*(BS^1) \cong L(e)$, a polynomial algebra on the degree 2 generator e . Essentially, there is only one way to construct a minimal

model for a space having such a cohomology algebra; $M(BS^1) \cong (L(e), d=0)$.

This observation leads to the following structure imposed on the model of the fibration $X \rightarrow XS^1 \rightarrow BS^1$;

$$(L(e), d=0) \rightarrow (L(e \oplus W), D) \rightarrow (L(W), d),$$

with $De = 0$ and $Dw = dw + \Omega$ where Ω is in the ideal generated by e .

The reader should consult [AH] or [O] for details and examples. Also, we note here that an extension of the above form may be realized by an almost free circle action on a space if $\dim H^*(L(e \oplus W), D) < \infty$ (see [O]).

Now we are in a position to construct our examples in an elementary fashion. Our construction is algebraic, but the statement above shows that we are essentially defining circle actions on the product of spheres $S^4 \times S^9$. Form the extension ($|$ denotes degree, $a \in \mathbb{Q}$),

$$L(e) \rightarrow (L(e, x, y, z), D_a) \rightarrow (L(x, y, z), d)$$

$$\begin{array}{lll} \text{with } |e| = 2 & D_a e = 0 = D_a x & |x| = 4, |y| = 7, |z| = 9 \\ de = 0 & D_a y = x^2 + ae^4 & dx = 0 \quad dy = x^2 \\ & D_a z = e^5 & dz = 0. \end{array}$$

It can easily be verified that D_a satisfies the conditions described earlier. Also, note that

$$(L(x, y), dx = 0, dy = x^2) \text{ and } (L(z), d = 0)$$

are the minimal models of S^4 and S^9 respectively, so the "fibre" models $S^4 \times S^9$.

Remark

(i) It can be computed directly that $\dim H^*(L(e, x, y, z), D_a) < \infty$. In fact, because $D_a z = e^5$, we see that $H^*(L(e)) \rightarrow H^*(L(e, x, y, z))$ is not injective, so this corresponds to the situation of an almost free circle action by the Fixed Point Condition.

(ii) Because the cohomology of $(L(e,x,y,z), D_a)$ is finite dimensional and there are a finite number of generators, Theorem 3 of [H2] implies that $H^*(L(e,x,y,z), D_a)$ satisfies Poincaré duality and has formal dimension 12.

(iii) The cohomology in dimension 6 has basis $\{[xe], [e^3]\}$ and the Poincaré quadratic form is equivalent over \mathbb{Q} to $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Hence, the form is a difference of squares with signature zero. By Theorem 13.2 of [S] we see that $(L(e,x,y,z), D_a)$ is a minimal model for a 12-dimensional manifold.

In the following, for concreteness, we will consider only the cases $a = 1, 2$. Let $D_1 = D$, $D_2 = \bar{D}$, $M = (L(e,x,y,z), D)$ and $N = (L(e,x,y,z), \bar{D})$.

Thoeorem.

$$(i) \quad M \otimes \mathbb{R} \cong N \otimes \mathbb{R}.$$

$$(ii) \quad M \not\cong N.$$

Proof.

(i) Define $\theta: M \otimes \mathbb{R} \rightarrow N \otimes \mathbb{R}$ by, $\theta e = e$, $\theta x = (1/\sqrt{2})x$, $\theta y = (1/2)y$ and $\theta z = z$. Note that,

$$\begin{aligned} \theta Dy &= \theta(x^2 + e^4) \\ &= [\theta(x)]^2 + [\theta(e)]^4 \\ &= \frac{1}{2} x^2 + e^4 \\ &= \frac{1}{2} \bar{D}y \\ &= \bar{D}(\frac{1}{2} y) \\ &= \bar{D} \theta y. \end{aligned}$$

It is obvious that $\theta D = \bar{D} \theta$ for the other generators of $M \otimes \mathbb{R}$; thus, because $M \otimes \mathbb{R}$ is freely generated by these generators, the relation holds globally. The same argument shows that θ is a global isomorphism because it is so on the vector spaces of indecomposables. Hence $M \otimes \mathbb{R}$ and $N \otimes \mathbb{R}$ are isomorphic DGA's.

(ii) We can try to define an isomorphism over \mathbb{Q} in the following way:
 $(\lambda, \sigma, \rho, \epsilon, \tau, \mu \in \mathbb{Q}) \quad \theta e = \lambda e, \quad \theta x = \sigma x + \rho e^2, \quad \theta y = \epsilon y, \quad \theta z = \tau z + \mu e y.$
 This θ is the most general possible. As before, the existence of θ hinges on whether or not the relation $\bar{D}\theta = \theta D$ is satisfied. We have,

$$\begin{aligned} \theta Dy &= \sigma^2 x^2 + 2\sigma\rho x e^2 + (\rho^2 + \lambda^4) e^4 \\ \bar{D}\theta y &= \epsilon x^2 + 2\epsilon e^4. \end{aligned}$$

This imposes the following conditions on the coefficients: $\epsilon = \sigma^2$,
 $\rho^2 + \lambda^4 = 2\epsilon$ and $2\sigma\rho = 0$. The last condition provides two cases:

1. $\sigma = 0$. By the first condition, this implies $\epsilon = 0$. But then the definition of θ implies $\theta y = 0$ which shows θ is not an isomorphism.

2. $\rho = 0$. The second condition shows that $\lambda^4 = 2\epsilon$. By the first condition, $\lambda^4 = 2\sigma^2$ or the equivalent $2 = (\lambda^2/\sigma)^2$. But $\lambda, \epsilon \in \mathbb{Q}$, so this would say that 2 is a rational square. This contradiction rules out the second case.

Therefore, since neither of the two cases can occur, it is impossible to find an isomorphism of M onto N (as DGA's). QED

Corollary.

Let X and Y denote 12-manifolds which have minimal models M and N respectively. Then X and Y have distinct rational homotopy types, but their real homotopy types are the same. In particular, the DeRham complexes, $\Omega(X)$ and $\Omega(Y)$, carry the same homotopical information.

Remark.

In Halperin's terminology, M and N are elliptic spaces with zero homotopy euler characteristic (i.e. the same number of odd and even generators). Hence by [H2], M and N are formal DGA's. Therefore, the spaces X and Y are formal (in our earlier sense) as well. This says

that, while the quality of formality itself exhibits no descent, formal spaces possess a rich descent theory.

3. Non-Existence of Descent for Rational Equivalences

Although, as we have seen, descent occurs for spaces, it is not at all clear that the same can be said for maps between spaces. By the categorical equivalence mentioned earlier it is sufficient to study the following question: Do there exist DGA maps of minimal DGA's $f, g: M \rightarrow N$ such that f and g are not (DGA)-homotopic, but f_R and g_R are (DGA)-homotopic? (Here, f_R and g_R are the natural extensions of f and g to $M \otimes_{\mathbb{Z}} \mathbb{R} \rightarrow N \otimes_{\mathbb{Z}} \mathbb{R}$.) We shall demonstrate, in this section, that this situation cannot occur if f and g are automorphisms of the minimal DGA M . We state this result as,

Theorem A.

Let M be a minimal DGA and suppose $f, g \in \text{Aut}(M)$. Then f is homotopic to g if and only if f_R is homotopic to g_R .

Because $f, g \in \text{Aut}(M)$, if $f \sim g$, then $fg^{-1} \sim 1$. Hence, Theorem A is equivalent to,

Theorem A'.

The automorphism f is homotopic to the identity if and only if f_R is homotopic to the identity.

Lemma.

It is sufficient to assume M is finitely generated.

Proof.

By ([DR]; Lemma 2.4), $f \simeq 1$ if and only if for all n , $f_n \simeq 1_n$ where the subscript n denotes the restriction of the maps to $M(n)$, the sub-DGA of M generated by elements of degree $\leq n$. Therefore, if we show Theorem A' holds for every finitely generated M , then it will hold for each f_n and consequently for f . QED

Proof of Theorem A'.

By the Lemma, we may assume without loss of generality that M is finitely generated. Hence, we may apply the results of Sullivan described in §1. (We note here that, when we refer to the algebraic groups below, we are actually referring to either the rational or real points of an appropriate algebraic group over the complex numbers.)

By extension of scalars, there is an injection of algebraic groups $\text{Aut}(M) \hookrightarrow \text{Aut}(M \otimes_{\mathbb{C}} \mathbb{R})$ which respects homotopy. Hence, we obtain the following commutative diagram of exact sequences,

$$\begin{array}{ccccc}
 U & \longrightarrow & \text{Aut}(M) & \longrightarrow & h - \text{Aut}(M) \\
 \downarrow & & \downarrow \emptyset & & \downarrow \theta \\
 V & \longrightarrow & \text{Aut}(M \otimes_{\mathbb{C}} \mathbb{R}) & \longrightarrow & h - \text{Aut}(M \otimes_{\mathbb{C}} \mathbb{R})
 \end{array}$$

where V consists of the automorphisms of $M \otimes_{\mathbb{C}} \mathbb{R}$ homotopic to the identity. In order to prove the theorem, we must show only that θ is injective. This is equivalent to showing that, if $\emptyset(f) \in V$, then $f \in U$. We will prove this statement by using the structure of the nilpotent Lie algebras $L(U)$ and $L(V)$. Recall

$$L(U) = \{[d, i] \mid i \in \text{Der}^{-1}(M)\}$$

$$L(V) = \{[d, i] \mid i \in \text{Der}^{-1}(M \otimes_{\mathbb{C}} \mathbb{R})\}$$

where $[d, i] = di + id$.

Lemma.

As vector spaces, $\text{Der}^{-1}(M) \otimes \mathbb{R} \cong \text{Der}^{-1}(M \otimes \mathbb{R})$.

Proof.

A derivation is determined by its effect on algebra generators. Let $\{x_\alpha\}$ be a finite set of algebra generators and $\{y_\beta\}$ a vector space basis for M . Define,

$$I = \{i_{\alpha\beta} \in \text{Der}^{-1}(M) \mid i_{\alpha\beta}(x_\alpha) = y_\beta, \quad i_{\alpha\beta}(x_\gamma) = 0\}.$$

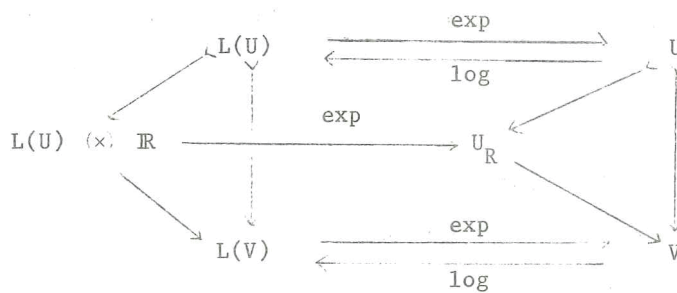
I forms a basis for the vector space of derivations. Because coefficients play no role here, we see that

$$\dim_Q \text{Der}^{-1}(M) = \dim_{\mathbb{R}} \text{Der}^{-1}(M \otimes \mathbb{R}).$$

Hence, $\dim_{\mathbb{R}}(\text{Der}^{-1}(M) \otimes \mathbb{R}) = \dim_{\mathbb{R}}(\text{Der}^{-1}(M \otimes \mathbb{R}))$ and the injection $\text{Der}^{-1}(M) \otimes \mathbb{R} \rightarrow \text{Der}^{-1}(M \otimes \mathbb{R})$ defined by $i_{\alpha\beta} \otimes 1 \mapsto i_{\alpha\beta}$ is an isomorphism.

QED Lemma.

Now, consider the commutative diagram,



where U_R is the real unipotent group associated to the real nilpotent Lie algebra $L(U) \otimes \mathbb{R}$ via the exponential map and the injection $U \rightarrow U_R$ is

defined via the logarithm and exponential. The map $U_R \rightarrow V$ is defined similarly.

As we have seen, the maps $[d, -]: \text{Der}^{-1}(M) \rightarrow L(U)$, $[d, -]: \text{Der}^{-1}(M \otimes \mathbb{R}) \rightarrow L(V)$ are surjective. Therefore, the Lemma and the commutativity of the diagram,

$$\begin{array}{ccc} \text{Der}^{-1}(M \otimes \mathbb{R}) & \longrightarrow & L(U \otimes \mathbb{R}) \\ \cong \downarrow & & \downarrow \\ \text{Der}^{-1}(M \otimes \mathbb{R}) & \longrightarrow & L(V) \end{array}$$

imply that $L(U) \otimes \mathbb{R} \rightarrow L(V)$ is surjective. Hence, via log and exp, $U_R \rightarrow V$ is surjective as well.

Now, $U_R \rightarrow V$ has unipotent kernel, so any rational point of V comes from a rational point of U_R (see [S], §6). But the rational points of U_R are precisely the points of U since, clearly, the rational points of $L(U) \otimes \mathbb{R}$ are exactly the points of $L(U)$.

Suppose $\emptyset(f) \in V$. Because $f \in \text{Aut}(M)$, $\emptyset(f)$ is a rational point of V and therefore comes from a point of U by the argument above. Now, \emptyset is injective, so the only point of $\text{Aut}(M)$ mapping to $\emptyset(f)$ is f itself. Hence, f must be an element of U . That is, f is homotopic to the identity, as was to be shown. QED Theorem A'.

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