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On t-Balancers, t-Balancing Numbers and Lucas t-Balancing Numbers

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Abstract: In this work, we determined the general terms of t-balancers, t-balancing numbers and Lucas t-balancing numbers in terms of balancing and Lucas-balancing numbers by solving the Pell equation $2x^2-y^2 = 2t^2+4t+1$ for some integer $t \geq 1$.

Keywords: balancing numbers, t-balancing numbers, Pell equation.
MSC2010: 11B37, 11B39, 11D09, 11D79.

1 Introduction

A positive integer n is called a balancing number ([2]) if the Diophantine equation

$$1+2+\cdots+(n-1)=(n+1)+(n+2)+\cdots+(n+r) \tag{1.1}$$

holds for some positive integer r which is called balancer corresponding to n. If n is a balancing number with balancer r, then from (1.1)

$$n^2=\frac{(n+r)(n+r+1)}{2} \quad \text{and} \quad r=\frac{-2n-1+\sqrt{8n^2+1}}{2}. \tag{1.2}$$

From (1.2), they noted that n is a balancing number if and only if n^2 is a triangular number and $8n^2+1$ is a perfect square. Though the definition of balancing numbers suggests that no balancing number should be less than 2. But from (1.2), Behera and Panda noted that $8(0)^2+1=1$ and $8(1)^2+1=3^2$ are perfect squares. So they accepted 0 and 1 to be balancing numbers. Let B_n denote the n^{th} balancing number. Then $B_0=0, B_1=1$, $B_2=6$ and $B_{n+1}=6B_n-B_{n-1}$ for $n \geq 2$.

Later Panda and Ray ([12]) defined that a positive integer n is called a cobalancing number if the Diophantine equation

$$1+2+\cdots+n=(n+1)+(n+2)+\cdots+(n+r) \tag{1.3}$$

holds for some positive integer r which is called cobalancer corresponding to n. If n is a cobalancing number with cobalancer r, then from (1.3)

$$n(n+1)=\frac{(n+r)(n+r+1)}{2} \quad \text{and} \quad r=\frac{-2n-1+\sqrt{8n^2+8n+1}}{2}. \tag{1.4}$$

From (1.4), they noted that n is a cobalancing number if and only if $n(n+1)$ is a triangular number and $8n^2+8n+1$ is a perfect square. Since $8(0)^2+8(0)+1=1$ is a perfect square, they accepted 0 to be a cobalancing number just like Behera and Panda accepted 0 and 1 to be balancing numbers. Let b_n denote the n^{th} cobalancing number. Then $b_0=b_1=0, b_2=2$ and $b_{n+1}=6b_n-b_{n-1}+2$ for $n\geq 2$.

It is clear from (1.1) and (1.3) that every balancing number is a cobalancer and every cobalancing number is a balancer, that is, $B_n=r_{n+1}$ and $R_n=b_n$ for $n\geq 1$, where R_n is the n^{th} the balancer and r_n is the n^{th} cobalancer. Since $R_n=b_n$, we get from (1.1) that

$$b_n=\frac{-2B_n-1+\sqrt{8B_n^2+1}}{2} \quad \text{and} \quad B_n=\frac{2b_n+1+\sqrt{8b_n^2+8b_n+1}}{2}. \tag{1.5}$$

Thus from (1.5), B_n is a balancing number if and only if $8B_n^2+1$ is a perfect square and b_n is a cobalancing number if and only if $8b_n^2+8b_n+1$ is a perfect square. Thus

$$C_n=\sqrt{8B_n^2+1} \quad \text{and} \quad c_n=\sqrt{8b_n^2+8b_n+1} \tag{1.6}$$

are integers which are called the n^{th} Lucas-balancing number and n^{th} Lucas-cobalancing number, respectively (Note that $C_0=c_0=1$).

Let $\alpha=1+\sqrt{2}$ and $\beta=1-\sqrt{2}$ be the roots of the characteristic equation for Pell numbers which are the numbers defined by $P_0=0, P_1=1$ and $P_n=2P_{n-1}+P_{n-2}$ for $n\geq 2$. Ray ([17]) derived some nice results on balancing numbers and Pell numbers his Phd thesis. Since x is a balancing number if and only if $8x^2+1$ is a perfect square, he set $8x^2+1=y^2$ for some integer $y\geq 1$. Then he get

$$y^2-8x^2=1 \tag{1.7}$$

which is a Pell equation ([1, 3, 9]). The fundamental solution of (1.7) is $(y_1,x_1)=(3,1)$. So $y_n+x_n\sqrt{8}=(3+\sqrt{8})^n$ for $n\geq 1$ and similarly $y_n-x_n\sqrt{8}=(3-\sqrt{8})^n$. Let $\gamma=3+\sqrt{8}$ and $\delta=3-\sqrt{8}$. Then he get $x_n=\frac{\gamma^n-\delta^n}{\gamma-\delta}$ which is the Binet formula for balancing numbers, that is, $B_n=\frac{\gamma^n-\delta^n}{\gamma-\delta}$. Since $\alpha^2=\gamma$ and $\beta^2=\delta$, he conclude that the Binet formula for balancing numbers is

$$B_n=\frac{\alpha^{2n}-\beta^{2n}}{4\sqrt{2}}.$$

Similarly

$$b_n=\frac{\alpha^{2n-1}-\beta^{2n-1}}{4\sqrt{2}}-\frac{1}{2}, C_n=\frac{\alpha^{2n}+\beta^{2n}}{2} \quad \text{and} \quad c_n=\frac{\alpha^{2n-1}+\beta^{2n-1}}{2}$$

for $n\geq 1$ (see also [4, 10, 11, 15]).

Balancing numbers and their generalizations have been investigated by several authors from many aspects. In [7], Liptai proved that there is no Fibonacci balancing number except 1 and in [8] he proved that there is no Lucas balancing number. In [19], Szalay considered the same problem and obtained some nice results by a different method. In [5], Kovács, Liptai and Olajos extended the concept of balancing numbers to the (a, b)-balancing numbers defined as follows: Let $a > 0$ and $b \geq 0$ be coprime integers. If

$$(a + b) + \cdots + (a(n-1) + b) = (a(n+1) + b) + \cdots + (a(n+r) + b)$$

for some integers $n, r \geq 1$, then $an + b$ is an (a, b)-balancing number. The sequence of (a, b)-balancing numbers is denoted by $B_m^{(a,b)}$ for $m \geq 1$. In [6], Liptai, Luca, Pintér and Szalay generalized the notion of balancing numbers to numbers defined as follows: Let $y, k, l \in \mathbb{Z}^+$ such that $y \geq 4$. Then a positive integer x with $x \leq y-2$ is called a (k, l)-power numerical center for y if $1^k + \cdots + (x-1)^k = (x+1)^l + \cdots + (y-1)^l$. They studied the number of solutions of the equation above and proved several effective and ineffective finiteness results for (k, l)-power numerical centers. For integers $k, x \geq 1$, let

$$\Pi_k(x) = x(x+1)\ldots(x+k-1).$$

Then it was proved in [5] that the equation $B_m = \Pi_k(x)$ for fixed integer $k \geq 2$ has only infinitely many solutions and for $k \in \{2, 3, 4\}$ all solutions were determined. In [21] Tengely, considered the case $k = 5$, that is, $B_m = x(x+1)(x+2)(x+3)(x+4)$ and proved that this Diophantine equation has no solution for $m \geq 0$ and $x \in \mathbb{Z}$. In [14], Panda, Komatsu and Davala considered the reciprocal sums of sequences involving balancing and Lucas-balancing numbers. In [16], Patel, Irmak and Ray considered incomplete balancing and Lucas-balancing numbers and in [18], Ray considered the sums of balancing and Lucas-balancing numbers by matrix methods.

Recently, almost balancing numbers first defined by Panda and Panda in [13]. A natural number n is called an almost balancing number if the Diophantine equation

$$|[(n+1) + (n+2) + \cdots + (n+r)] - [1 + 2 + \cdots + (n-1)]| = 1$$

holds for some positive integer r which is called the almost balancer. In [20], the first author derived some new results on almost balancing numbers, triangular numbers and square triangular numbers.

2 t-Balancing numbers.

In this section we try to determine the general terms of all t-balancers, t-balancing numbers and Lucas t-balancing numbers.

Let $t \geq 1$ be an integer. Then by considering (1.1), a positive integer n is called a t-balancing number if the Diophantine equation

$$1+2+\cdots+n-1=(n+1+t)+(n+2+t)+\cdots+(n+r+t) \tag{2.1}$$

holds for some positive integer r which is called t-balancer corresponding to n.

Let B_n^t denote the n^{th} t-balancing number and let R_n^t denote the n^{th} t-balancer. Then from (2.1), we get

$$R_n^t = \frac{-2B_n^t-2t-1+\sqrt{8(B_n^t)^2+8tB_n^t+(2t+1)^2}}{2} \quad \text{and} \tag{2.2}$$

$$B_n^t = \frac{2R_n^t+1+\sqrt{8(R_n^t)^2+8(t+1)R_n^t+1}}{2}. \tag{2.3}$$

From (2.2), we note that B_n^t is a t-balancing number if and only if $8(B_n^t)^2+8tB_n^t+(2t+1)^2$ is a perfect square. Thus

$$C_n^t = \sqrt{8(B_n^t)^2+8tB_n^t+(2t+1)^2} \tag{2.4}$$

is an integer which is called Lucas t-balancing number.

In order to determine the general terms of t-balancers, t-balancing numbers and Lucas t-balancing numbers, we have to determine the set of all (positive) integer solutions of the Pell equation

$$2x^2-y^2=2t^2+4t+1. \tag{2.5}$$

We see from (2.3) that R_n^t is a t-balancer if and only if $8(R_n^t)^2+8(t+1)R_n^t+1$ is a perfect square. So we set

$$8(R_n^t)^2+8(t+1)R_n^t+1=y^2 \tag{2.6}$$

for some integer $y \geq 1$. Then $2(2R_n^t+t+1)^2-y^2=2t^2+4t+1$ and putting

$$x=2R_n^t+t+1, \tag{2.7}$$

we get the Pell equation defined in (2.5).

Now let Δ be a non-square discriminant. Then the Δ-order O_Δ is defined to be the ring $O_\Delta=\{x+y\rho_\Delta : x,y\in\mathbb{Z}\}$, where $\rho_\Delta=\sqrt{\frac{\Delta}{4}}$ if $\Delta\equiv 0(\text{mod } 4)$ or $\frac{1+\sqrt{\Delta}}{2}$ if $\Delta\equiv 1(\text{mod } 4)$. So O_Δ is a subring of $\mathbb{Q}(\sqrt{\Delta})=\{x+y\sqrt{\Delta} : x,y\in\mathbb{Q}\}$. The unit group O_Δ^u is defined to be the group of units of the ring O_Δ.

Let $F(x,y) = ax^2 + bxy + cy^2$ be an indefinite integral quadratic form ([3]) of discriminant $\Delta = b^2 - 4ac$. Then we can rewrite $F(x,y) = ((xa + y\frac{b+\sqrt{\Delta}}{2})(xa + y\frac{b-\sqrt{\Delta}}{2}))/a$. So the module M_F of F is

$$M_F = \{xa + y\frac{b+\sqrt{\Delta}}{2} : x, y \in \mathbb{Z}\} \subset \mathbb{Q}(\sqrt{\Delta}).$$

Therefore we get $(u + v\rho_\Delta)(xa + y\frac{b+\sqrt{\Delta}}{2}) = x'a + y'\frac{b+\sqrt{\Delta}}{2}$, where

$$[x' \;\; y'] = \begin{cases} [x \;\; y]\begin{bmatrix} u - \frac{b}{2}v & av \\ -cv & u + \frac{b}{2}v \end{bmatrix} & \text{if } \Delta \equiv 0 (\text{mod } 4) \\ [x \;\; y]\begin{bmatrix} u + \frac{1-b}{2}v & av \\ -cv & u + \frac{1+b}{2}v \end{bmatrix} & \text{if } \Delta \equiv 1 (\text{mod } 4). \end{cases} \tag{2.8}$$

Let m be any integer and let Ω denote the set of all integer solutions of $F(x,y) = m$, that is, $\Omega = \{(x,y) : F(x,y) = m\}$. Then there is a bijection

$$\Psi : \Omega \to \{\gamma \in M_F : N(\gamma) = am\}.$$

The action of $O^u_{\Delta,1} = \{\alpha \in O^u_\Delta : N(\alpha) = 1\}$ on the set Ω is most interesting when Δ is a positive non-square since $O^u_{\Delta,1}$ is infinite. Therefore the orbit of each solution will be infinite and so the set Ω is either empty or infinite. Since $O^u_{\Delta,1}$ can be explicitly determined, the set Ω is satisfactorily described by the representation of such a list, called a set of representatives of the orbits. Let ε_Δ be the smallest unit of O_Δ that is greater than 1 and let $\tau_\Delta = \varepsilon_\Delta$ if $N(\varepsilon_\Delta) = 1$ or ε^2_Δ if $N(\varepsilon_\Delta) = -1$. Then every $O^u_{\Delta,1}$ orbit of integral solutions of $F(x,y) = m$ contains a solution $(x,y) \in \mathbb{Z} \times \mathbb{Z}$ such that $0 \leq y \leq U$, where $U = \left|\frac{am\tau_\Delta}{\Delta}\right|^{\frac{1}{2}}(1 - \frac{1}{\tau_\Delta})$ if $am > 0$ or $U = \left|\frac{am\tau_\Delta}{\Delta}\right|^{\frac{1}{2}}(1 + \frac{1}{\tau_\Delta})$ if $am < 0$. So for finding a set of representatives of the $O^u_{\Delta,1}$ orbits of integral solutions of $F(x,y) = m$, we must find for each integer y_0 in the range $0 \leq y_0 \leq U$, whether $\Delta y_0^2 + 4am$ is a perfect square or not since

$$ax_0^2 + bx_0y_0 + cy_0^2 = m \Leftrightarrow \Delta y_0^2 + 4am = (2ax_0 + by_0)^2. \tag{2.9}$$

If $\Delta y_0^2 + 4am$ is a perfect square, then from (2.9) we get

$$x_0 = \frac{-by_0 \pm \sqrt{\Delta y_0^2 + 4am}}{2a}.$$

So there is a set of representatives Rep $= \{[x_0 \;\; y_0]\}$. Consequently for the matrix M defined in (2.8), the set of all integer solutions of $F(x,y) = m$ is $\Omega = \{\pm(x,y) : [x \;\; y] = [x_0 \;\; y_0]M^n, n \in \mathbb{Z}\}$. If $\Delta y_0^2 + 4am$ is not a perfect square, then there are no integer solutions.

For the set of all integer solutions of (2.5), the indefinite form is $F(x,y) = 2x^2 - y^2$ of discriminant $\Delta = 8$. So $\tau_8 = 3 + 2\sqrt{2}$ and

$$M = \begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix} \tag{2.10}$$

from (2.8). Here we have two cases: $2t^2 + 4t + 1$ is a perfect square or not for $t \geq 1$.

2.1 Case 1: $2t^2 + 4t + 1$ is a perfect square.

In this case, we can give the following theorem first.

Theorem 2.1. *The quadratic Diophantine equation* $2t^2 + 4t + 1 = h^2$ *is satisfied for* $(t, h) = (P_{2n-1} - 1, c_n)$ *for* $n \geq 2$.

Proof. Let $2t^2 + 4t + 1 = h^2$ for some integer $h \geq 1$. Then $2(t+1)^2 - h^2 = 1$ and taking $t+1 = w$, we get the Pell equation $2w^2 - h^2 = 1$. The set of representatives is $\mathrm{Rep} = \{[\pm 1 \ \ 1]\}$ and in this case $[1 \ \ -1]M^n$ generates all integer solutions (w_n, h_n) for $n \geq 1$ for $M = \begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix}$. It can be easily seen that $M^n = \begin{bmatrix} C_n & 4B_n \\ 2B_n & C_n \end{bmatrix}$ for $n \geq 1$. So the set of all integer solutions of $2w^2 - h^2 = 1$ is $\{(-2B_n + C_n, 4B_n - C_n) : n \geq 1\}$. But we notice that $-2B_n + C_n = P_{2n-1}$ and $4B_n - C_n = c_n$. So the quadratic equation $2t^2 + 4t + 1 = h^2$ is satisfied for $(t, h) = (P_{2n-1} - 1, c_n)$. □

For the set of all integer solutions of (2.5) and the general terms of all t-balancers, t-balancing numbers and Lucas t-balancing numbers, we have two cases: $\#\mathrm{Rep} = 4$ or $\#\mathrm{Rep} > 4$.

Theorem 2.2. *If* $\#Rep = 4$, *then*

1. *the set of all integer solutions is* $\Omega = \{(x_{3n+1}, y_{3n+1}) : n \geq 0\} \cup \{(x_{3n-1}, y_{3n-1}), (x_{3n}, y_{3n}) : n \geq 1\}$, *where*

$$\begin{aligned}
(x_{3n+1}, y_{3n+1}) &= (2B_n + (t+1)C_n, (4t+4)B_n + C_n) \\
(x_{3n-1}, y_{3n-1}) &= (-2hB_n + hC_n, 4hB_n - hC_n) \\
(x_{3n}, y_{3n}) &= (-2B_n + (t+1)C_n, (4t+4)B_n - C_n).
\end{aligned}$$

2. *the general terms of t-balancers, t-balancing numbers and Lucas t-balancing numbers are*

$$\begin{aligned}
R^t_{3n} &= \frac{2B_n + (t+1)C_n - t - 1}{2} \\
R^t_{3n-1} &= \frac{-2B_n + (t+1)C_n - t - 1}{2} \\
R^t_{3n-2} &= \frac{-2hB_n + hC_n - t - 1}{2}
\end{aligned}$$

$$B^t_{3n} = \frac{(4t+6)B_n + (t+2)C_n - t}{2}$$
$$B^t_{3n-1} = \frac{(4t+2)B_n + tC_n - t}{2}$$
$$B^t_{3n-2} = \frac{2hB_n - t}{2}$$
$$C^t_{3n} = \sqrt{8(B^t_{3n})^2 + 8tB^t_{3n} + (2t+1)^2}$$
$$C^t_{3n-1} = \sqrt{8(B^t_{3n-1})^2 + 8tB^t_{3n-1} + (2t+1)^2}$$
$$C^t_{3n-2} = \sqrt{8(B^t_{3n-2})^2 + 8tB^t_{3n-2} + (2t+1)^2}$$

for $n \geq 1$.

Proof. **(1)** Let #Rep $= 4$. Then the set of representations is

$$\text{Rep} = \{[\pm(t+1) \quad 1], [\pm h \quad h]\},$$

and in this case

1. $[t+1 \quad 1]M^n$ generates all integer solutions (x_{3n+1}, y_{3n+1}) for $n \geq 0$,
2. $[t+1 \quad -1]M^n$ generates all integer solutions (x_{3n}, y_{3n}) for $n \geq 1$,
3. $[h \quad -h]M^n$ generates all integer solutions (x_{3n-1}, y_{3n-1}) for $n \geq 1$.

Thus the set of all integer solutions is $\Omega = \{(2B_n + (t+1)C_n, (4t+4)B_n + C_n) : n \geq 0\} \cup \{(-2hB_n + hC_n, 4hB_n - hC_n), (-2B_n + (t+1)C_n, (4t+4)B_n - C_n) : n \geq 1\}$.

(2) Note that $x = 2R^t_n + t + 1$ from (2.7). So

$$R^t_{3n} = \frac{2B_n + (t+1)C_n - t - 1}{2}$$

and from (2.3) and (2.6), we observe that

$$\begin{aligned} B^t_{3n} &= \frac{2R^t_{3n} + 1 + \sqrt{8(R^t_{3n})^2 + 8(t+1)R^t_{3n} + 1}}{2} \\ &= \frac{2B_n + (t+1)C_n - t - 1 + 1 + (4t+4)B_n + C_n}{2} \\ &= \frac{(4t+6)B_n + (t+2)C_n - t}{2} \end{aligned}$$

Thus

$$C^t_{3n} = \sqrt{8(B^t_{3n})^2 + 8tB^t_{3n} + (2t+1)^2}$$

by (2.4). The other cases can be proved similarly. □

Theorem 2.3. *If $\#Rep = 2k > 4$, then*

1. *the set of all integer solutions is*

$$\Omega = \{(x_{(2k-1)n+1}, y_{(2k-1)n+1}), (x_{(2k-1)n+i+1}, y_{(2k-1)n+i+1}), (x_{(2k-1)n+k}, y_{(2k-1)n+k}) : n \geq 0\} \cup \{(x_{(2k-1)n}, y_{(2k-1)n}), (x_{(2k-1)n-i}, y_{(2k-1)n-i}) : n \geq 1\},$$

where

$$\begin{aligned}
(x_{(2k-1)n+1}, y_{(2k-1)n+1}) &= (2B_n + (t+1)C_n, (4t+4)B_n + C_n) \\
(x_{(2k-1)n+i+1}, y_{(2k-1)n+i+1}) &= (2t_{2i}B_n + t_{2i-1}C_n, 4t_{2i-1}B_n + t_{2i}C_n) \\
(x_{(2k-1)n+k}, y_{(2k-1)n+k}) &= (2hB_n + hC_n, 4hB_n + hC_n) \\
(x_{(2k-1)n}, y_{(2k-1)n}) &= (-2B_n + (t+1)C_n, (4t+4)B_n - C_n) \\
(x_{(2k-1)n-i}, y_{(2k-1)n-i}) &= (-2t_{2i}B_n + t_{2i-1}C_n, 4t_{2i-1}B_n - t_{2i}C_n).
\end{aligned}$$

2. *the general terms of t-balancers, t-balancing numbers and Lucas t-balancing numbers are*

$$\begin{aligned}
R^t_{(2k-1)n} &= \frac{2B_n + (t+1)C_n - t - 1}{2} \\
R^t_{(2k-1)n-1} &= \frac{-2B_n + (t+1)C_n - t - 1}{2} \\
R^t_{(2k-1)n-i-1} &= \frac{-2t_{2i}B_n + t_{2i-1}C_n - t - 1}{2} \\
B^t_{(2k-1)n} &= \frac{(4t+6)B_n + (t+2)C_n - t}{2} \\
B^t_{(2k-1)n-1} &= \frac{(4t+2)B_n + tC_n - t}{2} \\
B^t_{(2k-1)n-i-1} &= \frac{(-2t_{2i} + 4t_{2i-1})B_n + (t_{2i-1} - t_{2i})C_n - t}{2} \\
C^t_{(2k-1)n} &= \sqrt{8(B^t_{(2k-1)n})^2 + 8tB^t_{(2k-1)n} + (2t+1)^2} \\
C^t_{(2k-1)n-1} &= \sqrt{8(B^t_{(2k-1)n-1})^2 + 8tB^t_{(2k-1)n-1} + (2t+1)^2} \\
C^t_{(2k-1)n-i-1} &= \sqrt{8(B^t_{(2k-1)n-i-1})^2 + 8tB^t_{(2k-1)n-i-1} + (2t+1)^2}
\end{aligned}$$

for $n \geq 1$ and

$$\begin{aligned}
R^t_{(2k-1)n+i} &= \frac{2t_{2i}B_n + t_{2i-1}C_n - t - 1}{2} \\
R^t_{(2k-1)n+k-1} &= \frac{2hB_n + hC_n - t - 1}{2}
\end{aligned}$$

$$\begin{aligned} B^t_{(2k-1)n+i} &= \frac{(2t_{2i}+4t_{2i-1})B_n+(t_{2i-1}+t_{2i})C_n-t}{2} \\ B^t_{(2k-1)n+k-1} &= \frac{6hB_n+2hC_n-t}{2} \\ C^t_{(2k-1)n+i} &= \sqrt{8(B^t_{(2k-1)n+i})^2+8tB^t_{(2k-1)n+i}+(2t+1)^2} \\ C^t_{(2k-1)n+k-1} &= \sqrt{8(B^t_{(2k-1)n+k-1})^2+8tB^t_{(2k-1)n+k-1}+(2t+1)^2} \end{aligned}$$

for $n \geq 0$,

where t_{2i-1} *and* t_{2i} *are positive integers such that* $2t^2_{2i-1}-t^2_{2i}=2t^2+4t+1$ *for* $1 \leq i \leq k-2, t+1<t_1<t_3<\cdots<t_{2k-5}<h$ *and* $1<t_2<t_4<\cdots<t_{2k-4}<h$.

Proof. **(1)** Let $\#\text{Rep}>4$. Then the set of representations is

$$\text{Rep}=\{[\pm(t+1)\ \ 1],[\pm t_{2i-1}\ \ t_{2i}],[\pm h\ \ h]\},$$

where t_{2i-1} and t_{2i} are positive integers such that $2t^2_{2i-1}-t^2_{2i}=2t^2+4t+1$ for $1 \leq i \leq k-2, t+1<t_1<t_3<\cdots<t_{2k-5}<h$ and $1<t_2<t_4<\cdots<t_{2k-4}<h$. Here

1. $[t+1\ \ 1]M^n$ generates all integer solutions $(x_{(2k-1)n+1}, y_{(2k-1)n+1})$ for $n \geq 0$,
2. $[t_{2i-1}\ \ t_{2i}]M^n$ generates all integer solutions $(x_{(2k-1)n+i+1}, y_{(2k-1)n+i+1})$ for $n \geq 0$,
3. $[h\ \ h]M^n$ generates all integer solutions $(x_{(2k-1)n+k}, y_{(2k-1)n+k})$ for $n \geq 0$,
4. $[t+1\ \ -1]M^n$ generates all integer solutions $(x_{(2k-1)n}, y_{(2k-1)n})$ for $n \geq 1$,
5. $[t_{2i-1}\ \ -t_{2i}]M^n$ generates all integer solutions $(x_{(2k-1)n-i}, y_{(2k-1)n-i})$ for $n \geq 1$.

Thus the set of all integer solutions is $\Omega=\{(2B_n+(t+1)C_n,(4t+4)B_n+C_n),(2t_{2i}B_n+t_{2i-1}C_n,4t_{2i-1}B_n+t_{2i}C_n),(2hB_n+hC_n,4hB_n+hC_n):n\geq 0\}\cup\{(-2B_n+(t+1)C_n,(4t+4)B_n-C_n),(-2t_{2i}B_n+t_{2i-1}C_n,4t_{2i-1}B_n-t_{2i}C_n):n\geq 1\}$.

(2) It can be proved as in the same way that Theorem 2.2 was proved. □

When $\#\text{Rep}=2k>4$, it is impossible to determine the set of representatives and #Rep in terms of t. For example in Table 1, the set of representatives is given for some values of t. That is why we assume that the set of representatives is Rep $=\{[\pm(t+1)\ \ 1],[\pm t_{2i-1}\ \ t_{2i}],[\pm h\ \ h]\}$, where t_{2i-1} and t_{2i} are positive integers such that $2t^2_{2i-1}-t^2_{2i}=2t^2+4t+1$ for $1\leq i\leq k-2, t+1<t_1<t_3<\cdots<t_{2k-5}<h$ and $1<t_2<t_4<\cdots<t_{2k-4}<h$.

Table 1.

t	set of representatives
984	$\{[\pm 985 \quad 1], [\pm 995 \quad 199], [\pm 1025 \quad 401],$ $[\pm 1267 \quad 1127], [\pm 1393 \quad 1393]\}$
5740	$\{[\pm 5741 \quad 1], [\pm 6001 \quad 2471], [\pm 6739 \quad 4991],$ $[\pm 6805 \quad 5167], [\pm 8119 \quad 8119]\}$
33460	$\{[\pm 33461 \quad 1], [\pm 35155 \quad 15247], [\pm 38935 \quad 28153],$ $[\pm 40409 \quad 32039], [\pm 47321 \quad 47321]\}$
195024	$\{[\pm 195025 \quad 1], [\pm 195083 \quad 6767], [\pm 195257 \quad 13457],$ $[\pm 197005 \quad 39401], [\pm 197743 \quad 46207], [\pm 199547 \quad 59737],$ $[\pm 202985 \quad 79601], [\pm 205933 \quad 93527], [\pm 205973 \quad 93703],$ $[\pm 207607 \quad 100657], [\pm 209405 \quad 107849], [\pm 211327 \quad 115103],$ $[\pm 219883 \quad 143623], [\pm 222425 \quad 151249], [\pm 227837 \quad 166583],$ $[\pm 236623 \quad 189503], [\pm 243355 \quad 205849], [\pm 243443 \quad 206057],$ $[\pm 246977 \quad 214303], [\pm 250747 \quad 222887], [\pm 254665 \quad 231601],$ $[\pm 271133 \quad 266377], [\pm 275807 \quad 275807]\}$

2.2 Case 2: $2t^2 + 4t + 1$ is not a perfect square.

In this case we again two cases: $\#\mathrm{Rep} = 2$ or $\#\mathrm{Rep} > 2$.

Theorem 2.4. *If $\#Rep = 2$, then*

1. *the set of all integer solutions is $\Omega = \{(x_{2n+1}, y_{2n+1}) : n \geq 0\} \cup \{(x_{2n}, y_{2n}) : n \geq 1\}$, where*

$$\begin{aligned}(x_{2n+1}, y_{2n+1}) &= (2B_n + (t+1)C_n, (4t+4)B_n + C_n)\\ (x_{2n}, y_{2n}) &= (-2B_n + (t+1)C_n, (4t+4)B_n - C_n).\end{aligned}$$

2. *the general terms of t-balancers, t-balancing numbers and Lucas t-balancing numbers are*

$$R^t_{2n} = \frac{2B_n + (t+1)C_n - t - 1}{2}$$

$$R^t_{2n-1} = \frac{-2B_n + (t+1)C_n - t - 1}{2}$$

$$B^t_{2n} = \frac{t(c_{n+1} - 1) + 2B_{n+1}}{2}$$

$$B^t_{2n-1} = \frac{t(c_{n+1}-1)+2B_n}{2}$$
$$C^t_{2n} = \sqrt{8(B^t_{2n})^2+8tB^t_{2n}+(2t+1)^2}$$
$$C^t_{2n-1} = \sqrt{8(B^t_{2n-1})^2+8tB^t_{2n-1}+(2t+1)^2}$$

for $n \geq 1$.

Proof. **(1)** Let $\#\text{Rep} = 2$. Then the set of representatives is

$$\text{Rep} = \{[\pm(t+1) \quad 1]\}.$$

In this case $[t+1 \quad 1]M^n$ generates all integer solutions (x_{2n+1}, y_{2n+1}) for $n \geq 0$ and $[t+1 \quad -1]M^n$ generates all integer solutions (x_{2n}, y_{2n}) for $n \geq 1$. Thus the set of all integer solutions is $\Omega = \{(2B_n+(t+1)C_n, (4t+4)B_n+C_n) : n \geq 0\} \cup \{(-2B_n+(t+1)C_n, (4t+4)B_n - C_n) : n \geq 1\}$.

(2) From (1), we observe that

$$R^t_{2n} = \frac{2B_n+(t+1)C_n-t-1}{2}.$$

Hence from (2.3) and (2.6), we get

$$\begin{aligned}
B^t_{2n} &= \frac{2R^t_{2n}+1+\sqrt{8(R^t_{2n})^2+8(t+1)R^t_{2n}+1}}{2}\\
&= \frac{2B_n+(t+1)C_n-t-1+1+(4t+4)B_n+C_n}{2}\\
&= \frac{t(4B_n+C_n-1)+6B_n+2C_n}{2}\\
&= \frac{t\left(4(\frac{\alpha^{2n}-\beta^{2n}}{4\sqrt{2}})+\frac{\alpha^{2n}+\beta^{2n}}{2}-1\right)+6(\frac{\alpha^{2n}-\beta^{2n}}{4\sqrt{2}})+2(\frac{\alpha^{2n}+\beta^{2n}}{2})}{2}\\
&= \frac{t\left(\alpha^{2n}(\frac{1}{\sqrt{2}}+\frac{1}{2})+\beta^{2n}(\frac{-1}{\sqrt{2}}+\frac{1}{2})-1\right)+\alpha^{2n}(\frac{3}{2\sqrt{2}}+1)+\beta^{2n}(\frac{-3}{2\sqrt{2}}+1)}{2}\\
&= \frac{t(\frac{\alpha^{2n+1}+\beta^{2n+1}}{2}-1)+2(\frac{\alpha^{2n+2}-\beta^{2n+2}}{4\sqrt{2}})}{2}\\
&= \frac{t(c_{n+1}-1)+2B_{n+1}}{2}.
\end{aligned}$$

Thus

$$C^t_{2n} = \sqrt{8(B^t_{2n})^2+8tB^t_{2n}+(2t+1)^2}$$

by (2.4). The others can be proved similarly. □

Theorem 2.5. *If $\#Rep = 2k > 2$, then*

1. *the set of all integer solutions is* $\Omega = \{(x_{2kn+1}, y_{2kn+1}), (x_{2kn+i+1}, y_{2kn+i+1}) : n \geq 0\} \cup \{(x_{2kn}, y_{2kn}), (x_{2kn-i}, y_{2kn-i}) : n \geq 1\}$, *where*

$$\begin{aligned}
(x_{2kn+1}, y_{2kn+1}) &= (2B_n + (t+1)C_n, (4t+4)B_n + C_n) \\
(x_{2kn+i+1}, y_{2kn+i+1}) &= (2t_{2i}B_n + t_{2i-1}C_n, 4t_{2i-1}B_n + t_{2i}C_n) \\
(x_{2kn}, y_{2kn}) &= (-2B_n + (t+1)C_n, (4t+4)B_n - C_n) \\
(x_{2kn-i}, y_{2kn-i}) &= (-2t_{2i}B_n + t_{2i-1}C_n, 4t_{2i-1}B_n - t_{2i}C_n).
\end{aligned}$$

2. *the general terms of t-balancers, t-balancing numbers and Lucas t-balancing numbers are*

$$\begin{aligned}
R^t_{2kn} &= \frac{2B_n + (t+1)C_n - t - 1}{2} \\
R^t_{2kn-1} &= \frac{-2B_n + (t+1)C_n - t - 1}{2} \\
R^t_{2kn-i-1} &= \frac{-2t_{2i}B_n + t_{2i-1}C_n - t - 1}{2} \\
B^t_{2kn} &= \frac{t(c_{n+1} - 1) + 2B_{n+1}}{2} \\
B^t_{2kn-1} &= \frac{t(c_{n+1} - 1) + 2B_n}{2} \\
B^t_{2kn-i-1} &= \frac{(-2t_{2i} + 4t_{2i-1})B_n + (t_{2i-1} - t_{2i})C_n - t}{2} \\
C^t_{2kn} &= \sqrt{8(B^t_{2kn})^2 + 8tB^t_{2kn} + (2t+1)^2} \\
C^t_{2kn-1} &= \sqrt{8(B^t_{2kn-1})^2 + 8tB^t_{2kn-1} + (2t+1)^2} \\
C^t_{2kn-i-1} &= \sqrt{8(B^t_{2kn-i-1})^2 + 8tB^t_{2kn-i-1} + (2t+1)^2}
\end{aligned}$$

for $n \geq 1$ and

$$\begin{aligned}
R^t_{2kn+i} &= \frac{2t_{2i}B_n + t_{2i-1}C_n - t - 1}{2} \\
B^t_{2kn+i} &= \frac{(2t_{2i} + 4t_{2i-1})B_n + (t_{2i-1} + t_{2i})C_n - t}{2} \\
C^t_{2kn+i} &= \sqrt{8(B^t_{2kn+i})^2 + 8tB^t_{2kn+i} + (2t+1)^2}
\end{aligned}$$

for $n \geq 0$,

where t_{2i-1} and t_{2i} are positive integers such that $2t_{2i-1}^2 - t_{2i}^2 = 2t^2 + 4t + 1$ for $1 \leq i \leq k-1, t+1 < t_1 < t_3 < \cdots < t_{2k-3}$ and $1 < t_2 < t_4 < \cdots < t_{2k-2}$.

Proof. **(1)** Let $\#\text{Rep} = 2k > 2$. Then the set of representatives is

$$\text{Rep} = \{[\pm(t+1) \quad 1], [\pm t_{2i-1} \quad t_{2i}]\},$$

where t_{2i-1} and t_{2i} are positive integers such that $2t_{2i-1}^2 - t_{2i}^2 = 2t^2 + 4t + 1$ for $1 \leq i \leq k-1, t+1 < t_1 < t_3 < \cdots < t_{2k-3}$ and $1 < t_2 < t_4 < \cdots < t_{2k-2}$. Here

1. $[t+1 \quad 1]M^n$ generates all integer solutions (x_{2kn+1}, y_{2kn+1}) for $n \geq 0$,
2. $[t+1 \quad -1]\ M^n$ generates all integer solutions (x_{2kn}, y_{2kn}) for $n \geq 1$,
3. $[t_{2i-1} \quad t_{2i}]M^n$ generates all integer solutions $(x_{2kn+i+1}, y_{2kn+i+1})$ for $n \geq 0$,
4. $[t_{2i-1} \quad -t_{2i}]M^n$ generates all integer solutions (x_{2kn-i}, y_{2kn-i}) for $n \geq 1$.

Thus the set of all integer solutions is $\Omega = \{(2B_n+(t+1)C_n, (4t+4)B_n+C_n), (2t_{2i}B_n+t_{2i-1}C_n, 4t_{2i-1}B_n+t_{2i}C_n) : n \geq 0\} \cup \{(-2B_n+(t+1)C_n, (4t+4)B_n-C_n), (-2t_{2i}B_n+t_{2i-1}C_n, 4t_{2i-1}B_n - t_{2i}C_n) : n \geq 1\}$.

(2) It can be proved as in the same way that Theorem 2.4 was proved. □

Again when $\#\text{Rep} = 2k > 2$, it is impossible to determine the set of representatives and #Rep in terms of t. For example in Table 2, the set of representatives is given for some values of t.

Table 2.

t	set of representatives
11	$\{[\pm 12 \quad 1], [\pm 16 \quad 15]\}$
28	$\{[\pm 29 \quad 1], [\pm 41 \quad 41]\}$
43	$\{[\pm 44 \quad 1], [\pm 46 \quad 19], [\pm 56 \quad 49]\}$
57	$\{[\pm 58 \quad 1], [\pm 62 \quad 31], [\pm 74 \quad 65]\}$
36	$\{[\pm 37 \quad 1], [\pm 41 \quad 25], [\pm 43 \quad 31], [\pm 47 \quad 41]\}$
53	$\{[\pm 54 \quad 1], [\pm 56 \quad 21], [\pm 60 \quad 37], [\pm 70 \quad 63]\}$

That is why we assume that the set of representatives is $\text{Rep} = \{[\pm(t+1) \quad 1], [\pm t_{2i-1} \quad t_{2i}]\}$, where t_{2i-1} and t_{2i} are positive integers such that $2t_{2i-1}^2 - t_{2i}^2 = 2t^2 + 4t + 1$ for $1 \leq i \leq k-1, t+1 < t_1 < t_3 < \cdots < t_{2k-3}$ and $1 < t_2 < t_4 < \cdots < t_{2k-2}$.

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