Libertas Mathematica (new series)
Volume 37(2017), No. 1, 45-64

Hilfer and Hilfer-Hadamard Fractional Differential Equations with Random Effects

Saïd Abbas, Mouffak Benchohra and Naima Hamidi

Abstract: This paper deals with some existence and Ulam stability results for some functional differential equations of Hilfer and Hilfer-Hadamard type. An application is made of Itoh's random fixed point theorem for the existence of random solutions. Next we prove that our problems are generalized Ulam-Hyers-Rassias stable.

Keywords: Functional Random differential equation, left-sided mixed Riemann-Liouville integral of fractional order, left-sided mixed Hadamard integral of fractional order, Hilfer fractional derivative, Hilfer-Hadamard fractional derivative, existence, Ulam stability, random solution, fixed point.

MSC2010: 26A33

1 Introduction

The fractional calculus deals with extensions of derivatives and integrals to non-integer orders. It represents a powerful tool in applied mathematics to study many problems from different fields of science and engineering, with many break-through results found in mathematical physics, finance, hydrology, biophysics, thermodynamics, control theory, statistical mechanics, astrophysics, cosmology and bioengineering [14, 29]. For some fundamental results in the theory of fractional calculus and fractional differential equations we refer the reader to the monographs of Abbas *et al.* [7, 8, 9], Samko *et al.* [28], Kilbas *et al.* [22] and Zhou [33, 34], the papers by Abbas *et al.* [1, 4, 5, 10, 11], and the references therein.

The stability of functional equations was originally raised by Ulam [31]). next by Hyers [15]. Thereafter, this type of stability is called the Ulam-Hyers stability. In 1978, Rassias [25] provided a remarkable generalization of the Ulam-Hyers stability of mappings by considering variables. The concept of stability for a functional equation arises when we replace the functional equation by an inequality which acts as a perturbation of the equation. Considerable attention has been given to the study of the Ulam-Hyers and Ulam-Hyers-Rassias stability of all kinds of functional equations; one can see the monographs of [9, 17], and the papers of Abbas *et al.* [1, 2, 3, 4, 6, 10, 11], Petru *et al.* [23], and Rus [26, 27] discussed the Ulam-Hyers stability for operatorial equations and inclusions. More details from historical point of view, and recent developments of such stabilities are reported in [18, 26].

Recently, considerable attention has been given to the existence of solutions of initial and boundary value problems for fractional differential equations with Hilfer fractional derivative; see [12, 13, 14, 19, 30, 32]. Motivated by the above papers, in this article we discuss the existence and the Ulam stability of solutions for the following problem of Random Hilfer fractional differential equations of the form

$$\begin{cases} (D_0^{\alpha,\beta}u)(t,w) = f(t,u(t,w),w);\ t \in I := [0,T], \\ \\ (I_0^{1-\gamma}u)(t,w)|_{t=0} = \phi(w), \end{cases} \qquad w \in \Omega, \tag{1.1}$$

where $\alpha \in (0,1),\ \beta \in [0,1],\ \gamma = \alpha+\beta-\alpha\beta,\ T > 0,\ (\Omega,\mathcal{A})$ is a measurable space, $\phi : \Omega \to \mathbb{R}$ is a measurable function, $f : I \times \mathbb{R} \times \Omega \to \mathbb{R}$ is a given function, $I_0^{1-\gamma}$ is the left-sided mixed Riemann-Liouville integral of order $1-\gamma$, and $D_0^{\alpha,\beta}$ is the Hilfer fractional derivative of order α and type β.

Next, we consider the following problem of random Hilfer-Hadamard fractional differential equations of the form

$$\begin{cases} ({}^HD_1^{\alpha,\beta}u)(t,w) = g(t,u(t,w),w);\ t \in [1,T], \\ \\ ({}^HI_1^{1-\gamma}u)(1,w) = \phi_0(w), \end{cases} \qquad w \in \Omega, \tag{1.2}$$

where $\alpha \in (0,1),\ \beta \in [0,1],\ \gamma = \alpha+\beta-\alpha\beta,\ T > 1,\ \phi_0 : \Omega \to \mathbb{R}$ is a measurable function, $g : [1,T] \times \mathbb{R} \times \Omega \to \mathbb{R}$ is a given function, ${}^HI_1^{1-\gamma}$ is the left-sided mixed Hadamard integral of order $1-\gamma$, and ${}^HD_1^{\alpha,\beta}$ is the Hilfer-Hadamard fractional derivative of order α and type β.

The present paper initiates the Ulam stability for random differential equations involving Hilfer and Hilfer-Hadamard fractional derivatives.

2 Preliminaries

Let C be the Banach space of all continuous functions v from I into $\mathbb{R}$ with the supremum (uniform) norm

$$\|v\|_\infty := \sup_{t\in I} |v(t)|.$$

As usual, $AC(I)$ denotes the space of absolutely continuous functions from I into $\mathbb{R}$. We denote by $AC^1(I)$ the space defined by

$$AC^1(I) := \{w : I \to \mathbb{R} : \frac{d}{dt}w(t) \in AC(I)\}.$$

By $L^1(I)$, we denote the space of Lebesgue-integrable functions $v : I \to \mathbb{R}$ with the norm

$$\|v\|_1 = \int_0^T |v(t)|dt.$$

Let $L^\infty(I)$ be the Banach space of measurable functions $u : I \to \mathbb{R}$ which are essentially bounded, equipped with the norm

$$\|u\|_{L^\infty} = \inf\{c > 0 : |u(t)| \leq c,\ a.e.\ t \in I\}.$$

By $C_\gamma(I)$ and $C^1_\gamma(I)$, we denote the weighted spaces of continuous functions defined by

$$C_\gamma(I) = \{w : (0, T] \to \mathbb{R} : t^{1-\gamma}w(t) \in C\},$$

with the norm

$$\|w\|_{C_\gamma} := \sup_{t\in I} |t^{1-\gamma}w(t)|,$$

and

$$C^1_\gamma(I) = \{w \in C : \frac{dw}{dt} \in C_\gamma\},$$

with the norm

$$\|w\|_{C^1_\gamma} := \|w\|_\infty + \|w'\|_{C_\gamma}.$$

Throughout this paper, we denote $\|w\|_{C_\gamma}$ by $\|w\|_C$.

Definition 2.1. A function $T : \Omega \times \mathbb{R} \to \mathbb{R}$ is called jointly measurable if $T(\cdot, u)$ is measurable for all $u \in \mathbb{R}$ and $T(w, \cdot)$ is continuous for all $w \in \Omega$.

Definition 2.2. A function $f : I \times \mathbb{R} \times \Omega \to \mathbb{R}$ is called random Carathéodory if the following conditions are satisfied:

(i) The map $(t, w) \to f(t, u, w)$ is jointly measurable for all $u \in \mathbb{R}$, and

(ii) The map $u \to f(t, u, w)$ is continuous for all $t \in I$ and $w \in \Omega$.

Let E be a Banach space and $T : \Omega \times E \to E$ be a mapping. Then T is called a random operator if $T(w, u)$ is measurable in w for all $u \in E$ and it expressed as $T(w)u = T(w, u)$. In this case we also say that $T(w)$ is a random operator on E. A random operator $T(w)$ on E is called continuous (resp. compact, totally bounded and completely continuous) if $T(w,u)$ is continuous (resp. compact, totally bounded and completely continuous) in u for all $w \in \Omega$. The details of completely continuous random operators in Banach spaces and their properties appear in Itoh [16].

Definition 2.3. Let $\mathcal{P}(Y)$ be the family of all nonempty subsets of Y and C be a mapping from Ω into $\mathcal{P}(Y)$. A mapping $T : \{(w,y) : w \in \Omega,\ y \in C(w)\} \to Y$ is called random operator with stochastic domain C if C is measurable (i.e., for all closed $A \subset Y$, $\{w \in \Omega, C(w) \cap A \neq \emptyset\}$ is measurable) and for all open $D \subset Y$ and all $y \in Y$, $\{w \in \Omega : y \in C(w), T(w,y) \in D\}$ is measurable. T will be called continuous if every $T(w)$ is continuous. For a random operator T, a mapping $y : \Omega \to Y$ is called random (stochastic) fixed point of T if for almost all $w \in \Omega$, $y(w) \in C(w)$ and $T(w)y(w) = y(w)$ and for all open $D \subset Y$, $\{w \in \Omega : y(w) \in D\}$ is measurable.

Now, we give some results and properties of fractional calculus.

Definition 2.4. [8, 22, 28] The left-sided mixed Riemann-Liouville integral of order $r > 0$ of a function $w \in L^1(I)$ is defined by

$$(I_0^r w)(t) = \frac{1}{\Gamma(r)} \int_0^t (t-s)^{r-1} w(s)ds;\ for\ a.e.\ t \in I,$$

where $\Gamma(\cdot)$ is the (Euler's) Gamma function defined by

$$\Gamma(\xi) = \int_0^\infty t^{\xi-1} e^{-t} dt;\ \xi > 0.$$

Notice that for all $r, r_1, r_2 > 0$ and each $w \in C$, we have $I_0^r w \in C$, and

$$(I_0^{r_1} I_0^{r_2} w)(t) = (I_0^{r_1+r_2} w)(t);\ for\ a.e.\ t \in I.$$

Definition 2.5. [8, 22, 28] The Riemann-Liouville fractional derivative of order $r \in (0,1]$ of a function $w \in L^1(I)$ is defined by

$$\begin{aligned}(D_0^r w)(t) &= \left(\frac{d}{dt} I_0^{1-r} w\right)(t) \\ &= \frac{1}{\Gamma(1-r)} \frac{d}{dt} \int_0^t (t-s)^{-r} w(s)ds;\ for\ a.e.\ t \in I.\end{aligned}$$

Let $r \in (0,1]$, $\gamma \in [0,1)$ and $w \in C_{1-\gamma}(I)$. Then the following expression leads to the left inverse operator as follows.

$$(D_0^r I_0^r w)(t) = w(t);\ for\ all\ t \in (0,T].$$

Moreover, if $I_0^{1-r}w \in C^1_{1-\gamma}(I)$, then the following composition is proved in [28]

$$(I_0^r D_0^r w)(t) = w(t) - \frac{(I_0^{1-r}w)(0^+)}{\Gamma(r)} t^{r-1};\ for\ all\ t \in (0,T].$$

Definition 2.6. [8, 22, 28] The Caputo fractional derivative of order $r \in (0,1]$ of a function $w \in L^1(I)$ is defined by

$$\begin{aligned}({}^cD_0^r w)(t) &= \left(I_0^{1-r}\frac{d}{dt}w\right)(t)\\ &= \frac{1}{\Gamma(1-r)}\int_0^t (t-s)^{-r}\frac{d}{ds}w(s)ds;\ for\ a.e.\ t \in I.\end{aligned}$$

In [14], R. Hilfer studied applications of a generalized fractional operator having the Riemann-Liouville and the Caputo derivatives as specific cases (see also [19, 30].

Definition 2.7. (Hilfer derivative). Let $\alpha \in (0,1)$, $\beta \in [0,1]$, $w \in L^1(I)$, $I_0^{(1-\alpha)(1-\beta)}w \in AC^1(I)$. The Hilfer fractional derivative of order α and type β of w is defined as

$$(D_0^{\alpha,\beta}w)(t) = \left(I_0^{\beta(1-\alpha)}\frac{d}{dt}I_0^{(1-\alpha)(1-\beta)}w\right)(t);\ for\ a.e.\ t \in I. \tag{2.1}$$

Properties. Let $\alpha \in (0,1)$, $\beta \in [0,1]$, $\gamma = \alpha + \beta - \alpha\beta$, and $w \in L^1(I)$.

1. The operator $(D_0^{\alpha,\beta}w)(t)$ can be written as

$$(D_0^{\alpha,\beta}w)(t) = \left(I_0^{\beta(1-\alpha)}\frac{d}{dt}I_0^{1-\gamma}w\right)(t) = \left(I_0^{\beta(1-\alpha)}D_0^{\gamma}w\right)(t);\ for\ a.e.\ t \in I.$$

Moreover, the parameter γ satisfies

$$\gamma \in (0,1],\ \gamma \geq \alpha,\ \gamma > \beta,\ 1-\gamma < 1-\beta(1-\alpha).$$

2. The generalization (2.1) for $\beta = 0$, coincides with the Riemann-Liouville derivative and for $\beta = 1$ with the Caputo derivative.

$$D_0^{\alpha,0} = D_0^{\alpha},\ and\ D_0^{\alpha,1} = {}^cD_0^{\alpha}.$$

3. If $D_0^{\beta(1-\alpha)}w$ exists and in $L^1(I)$, then

$$(D_0^{\alpha,\beta}I_0^{\alpha}w)(t) = (I_0^{\beta(1-\alpha)}D_0^{\beta(1-\alpha)}w)(t);\ for\ a.e.\ t \in I.$$

Furthermore, if $w \in C_\gamma(I)$ and $I_0^{1-\beta(1-\alpha)}w \in C^1_\gamma(I)$, then

$$(D_0^{\alpha,\beta}I_0^{\alpha}w)(t) = w(t);\ for\ a.e.\ t \in I.$$

4. If $D_0^\gamma w$ exists and in $L^1(I)$, then

$$(I_0^\alpha D_0^{\alpha,\beta} w)(t) = (I_0^\gamma D_0^\gamma w)(t) = w(t) - \frac{I_0^{1-\gamma}(0^+)}{\Gamma(\gamma)} t^{\gamma-1}; \ for\ a.e.\ t \in I.$$

Lemma 2.8. *Let $h \in C_\gamma(I)$. Then the linear Cauchy problem*

$$\begin{cases} (D_0^{\alpha,\beta} u)(t) = h(t); \ t \in I, \\ \\ (I_0^{1-\gamma} u)(t)|_{t=0} = \phi, \end{cases}$$

has a unique solution $u \in L^1(I)$ given by

$$u(t) = \frac{\phi}{\Gamma(\gamma)} t^{\gamma-1} + (I_0^\alpha h)(t).$$

From the above lemma, we concluded the following lemma

Lemma 2.9. *Let $f : I \times \mathbb{R} \times \Omega \to \mathbb{R}$ be such that $f(\cdot, u(\cdot, w), w) \in C_\gamma$ for all $w \in \Omega$, and any $u(w) \in C_\gamma$. Then problem (1.1) is equivalent to the problem of the solutions of the integral equation*

$$u(t, w) = \frac{\phi(w)}{\Gamma(\gamma)} t^{\gamma-1} + (I_0^\alpha f(\cdot, u(\cdot, w), w))(t); \ w \in \Omega.$$

Now, we consider the Ulam stability for the problem (1.1). Let $\epsilon > 0$ and $\Phi : I \times \Omega \to [0, \infty)$ be a continuous function. We consider the following inequalities

$$|(D_0^{\alpha,\beta} u)(t, w) - f(t, u(t, w), w)| \le \epsilon; \ t \in I, \ w \in \Omega. \tag{2.2}$$

$$|(D_0^{\alpha,\beta} u)(t, w) - f(t, u(t, w), w)| \le \Phi(t, w); \ t \in I, \ w \in \Omega. \tag{2.3}$$

$$|(D_0^{\alpha,\beta} u)(t, w) - f(t, u(t, w), w)| \le \epsilon\Phi(t, w); \ t \in I, \ w \in \Omega. \tag{2.4}$$

Definition 2.10. *[8, 26]* The problem *(1.1)* is Ulam-Hyers stable if there exists a real number $c_f > 0$ such that for each $\epsilon > 0$ and for each random solution $u : \Omega \to C_\gamma$ of the inequality *(2.2)* there exists a random solution $v : \Omega \to C_\gamma$ of *(1.1)* with

$$|u(t, w) - v(t, w)| \le \epsilon c_f; \ t \in I, \ w \in \Omega.$$

Definition 2.11. *[8, 26]* The problem *(1.1)* is generalized Ulam-Hyers stable if there exists $c_f : C([0, \infty), [0, \infty))$ with $c_f(0) = 0$ such that for each $\epsilon > 0$ and for each random solution $u : \Omega \to C_\gamma$ of the inequality *(2.2)* there exists a random solution $v : \Omega \to C_\gamma$ of *(1.1)* with

$$|u(t, w) - v(t, w)| \le c_f(\epsilon); \ t \in I, \ w \in \Omega.$$

Definition 2.12. *[8, 26]* The problem *(1.1)* is Ulam-Hyers-Rassias stable with respect to Φ if there exists a real number $c_{f,\Phi} > 0$ such that for each $\epsilon > 0$ and for each random solution $u : \Omega \to C_\gamma$ of the inequality *(2.4)* there exists a random solution $v : \Omega \to C_\gamma$ of *(1.1)* with

$$|u(t,w) - v(t,w)| \leq \epsilon c_{f,\Phi}\Phi(t,w);\ t \in I,\ w \in \Omega.$$

Definition 2.13. *[8, 26]* The problem *(1.1)* is generalized Ulam-Hyers-Rassias stable with respect to Φ if there exists a real number $c_{f,\Phi} > 0$ such that for each random solution $u : \Omega \to C_\gamma$ of the inequality *(2.3)*, there exists a random solution $v : \Omega \to C_\gamma$ of *(1.1)* with

$$|u(t,w) - v(t,w)| \leq c_{f,\Phi}\Phi(t,w);\ t \in I,\ w \in \Omega.$$

Remark 2.14. It is clear that

(i) Definition 2.10 $\Rightarrow$ Definition 2.11,

(ii) Definition 2.12 $\Rightarrow$ Definition 2.13,

(iii) Definition 2.12 for $\Phi(\cdot,\cdot) = 1 \ \Rightarrow$ Definition 2.10.

One can have similar remarks for the inequalities (2.2) and (2.4).

In the sequel, we employ the following random fixed point theorem.

Theorem 2.15. *(Itoh [16]) Let X be a non-empty, closed convex bounded subset of the separable Banach space E and let $N : \Omega \times X \to X$ be a compact and continuous random operator. Then the random equation $N(w)u = u$ has a random solution.*

3 Hilfer fractional random differential equations

In this section, we are concerned with the existence and the Ulam-Hyers-Rassias stability for problem (1.1). Let us start by defining what we mean by a random solution of the problem (1.1).

Definition 3.1. By a random solution of the problem (1.1) we mean a measurable function $u : \Omega \to C_\gamma$ that satisfies the condition $(I_0^{1-\gamma}u)(0^+, w) = \phi(w)$, and the equation $(D_0^{\alpha,\beta}u)(t,w) = f(t, u(t,w), w)$ on $I \times \Omega$.

The following hypotheses will be used in the sequel.

(H_1) The function f is random Carathéodory on $I \times \mathbb{R} \times \Omega$,

(H_2) There exist a measurable and bounded function $p : \Omega \to L^\infty(I, [0, \infty))$, such that

$$|f(t, u, w)| \leq \frac{p(t, w)|u|}{1 + |u|}; \; for\ a.e.\ t \in I,\ and\ each\ u \in \mathbb{R},\ w \in \Omega.$$

Set

$$p^* = \sup_{w \in \Omega} \|p(w)\|_{L^\infty},\ and\ \phi^* = \sup_{w \in \Omega} |\phi(w)|.$$

Now, we shall prove the following theorem concerning the existence of random solutions of problem (1.1).

Theorem 3.2. *Assume that the hypotheses (H_1) and (H_2) hold. Then the problem (1.1) has at least one random solution defined on $I \times \Omega$.*

Proof. Define a mapping $N : \Omega \times C_\gamma \to C_\gamma$ by:

$$(N(w)u)(t) = \frac{\phi(w)}{\Gamma(\gamma)} t^{\gamma-1} + \int_0^t (t - s)^{\alpha-1} \frac{f(s, u(s, w), w)}{\Gamma(\alpha)} ds. \tag{3.1}$$

The map ϕ is measurable for all $w \in \Omega$. Again, as the indefinite integral is continuous on I, then $N(w)$ defines a mapping $N : \Omega \times C_\gamma \to C_\gamma$. Thus u is a random solution for the problem (1.1) if and only if $u = N(w)u$.
Next, for any $u \in C_\gamma$, and each $t \in I$ and $w \in \omega$, we have

$$\begin{aligned}
|t^{1-\gamma}(N(w)u)(t)| &\leq \frac{|\phi(w)|}{\Gamma(\gamma)} + \frac{t^{1-\gamma}}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1} |f(s, u(s, w), w)| ds \\
&\leq \frac{|\phi(w)|}{\Gamma(\gamma)} + \frac{t^{1-\gamma}}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1} p(s, w) ds \\
&\leq \frac{\phi^*}{\Gamma(\gamma)} + \frac{p^* T^{1-\gamma}}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1} ds \\
&\leq \frac{\phi^*}{\Gamma(\gamma)} + \frac{p^* T^{1-\gamma+\alpha}}{\Gamma(1 + \alpha)}.
\end{aligned}$$

Thus

$$\|N(w)u\|_C \leq \frac{\phi^*}{\Gamma(\gamma)} + \frac{p^* T^{1-\gamma+\alpha}}{\Gamma(1 + \alpha)} := R. \tag{3.2}$$

This proves that $N(w)$ transforms the ball $B_R := B(0, R) = \{u \in C_\gamma : \|u\|_C \leq R\}$ into itself. We shall show that the operator $N : \Omega \times B_R \to B_R$ satisfies all the assumptions of Theorem 2.15. The proof will be given in several steps.

Step 1. $N(w)$ is a random operator on $\Omega \times B_R$ into B_R.
Since $f(t,u,w)$ is random Carathéodory, the map $w \to f(t,u,w)$ is measurable in view of Definition 2.1. Similarly, the product $(t-s)^{\alpha-1}f(s,u(s,w),w)$ of a continuous and a measurable function is again measurable. Further, the integral is a limit of a finite sum of measurable functions, therefore, the map

$$w \mapsto \frac{\phi(w)}{\Gamma(\gamma)}t^{\gamma-1} + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}f(s,u(s,w),w)ds,$$

is measurable. As a result, $N(w)$ is a random operator on $\Omega \times B_R$ into B_R.

Step 2. $N(w)$ *is continuous in* u.
Let $\{u_n\}_{n\in\mathbb{N}}$ be a sequence such that $u_n \to u$ in B_R. Then, for each $t \in I$, and $w \in \Omega$, we have

$$\begin{aligned}&|t^{1-\gamma}(N(w)u_n)(t) - t^{1-\gamma}(N(w)u)(t)|\\ &\leq \frac{t^{1-\gamma}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|f(s,u_n(s,w),w) - f(s,u(s,w),w)|ds. \qquad (3.3)\end{aligned}$$

Since $u_n \to u$ as $n \to \infty$ and f is random Carathéeodory, then by the Lebesgue dominated convergence theorem, equation (3.3) implies

$$\|N(w)u_n - N(w)u\|_C \to 0 \quad \text{as } n \to \infty.$$

Step 3. $N(w)B_R$ *is uniformly bounded.*
This is clear since $N(w)B_R \subset B_R$ and B_R is bounded.

Step 4. $N(w)B_R$ *is equicontinuous.*
Let $t_1, t_2 \in I$, $t_1 < t_2$ and let $u \in B_R$. Then, for each $w \in \Omega$, we have

$$\begin{aligned}&|t_2^{1-\gamma}(N(w)u)(t_2) - t_1^{1-\gamma}(N(w)u)(t_1)|\\
&\leq \left|t_2^{1-\gamma}\int_0^{t_2}(t_2-s)^{\alpha-1}\frac{f(s,u(s,w),w)}{\Gamma(\alpha)}ds - t_1^{1-\gamma}\int_0^{t_1}(t_1-s)^{\alpha-1}\frac{f(s,u(s,w),w)}{\Gamma(\alpha)}ds\right|\\
&\leq t_2^{1-\gamma}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}\frac{|f(s,u(s,w),w)|}{\Gamma(\alpha)}ds\\
&\quad + \int_0^{t_1}|t_2^{1-\gamma}(t_2-s)^{\alpha-1} - t_1^{1-\gamma}(t_1-s)^{\alpha-1}|\frac{|f(s,u(s,w),w)|}{\Gamma(\alpha)}ds\\
&\leq t_2^{1-\gamma}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}\frac{p(s,w)}{\Gamma(\alpha)}ds\\
&\quad + \int_0^{t_1}|t_2^{1-\gamma}(t_2-s)^{\alpha-1} - t_1^{1-\gamma}(t_1-s)^{\alpha-1}|\frac{p(s.w)}{\Gamma(\alpha)}ds.\end{aligned}$$

Thus, we get

$$|t_2^{1-\gamma}(N(w)u)(t_2) - t_1^{1-\gamma}(N(w)u)(t_1)| \leq \frac{p^* T^{1-\gamma+\alpha}}{\Gamma(1+\alpha)}(t_2 - t_1)^\alpha$$
$$+ \frac{p^*}{\Gamma(\alpha)} \int_0^{t_1} |t_2^{1-\gamma}(t_2 - s)^{\alpha-1} - t_1^{1-\gamma}(t_1 - s)^{\alpha-1}|ds.$$

As $t_1 \longrightarrow t_2$, the right-hand side of the above inequality tends to zero.

As a consequence of steps 1 to 4 together with the Arzelá-Ascoli theorem, we can conclude that $N : \Omega \times B_R \to B_R$ is continuous and compact. From an application of Theorem 2.15, we deduce that the operator equation $N(w)u = u$ has a random solution. This implies that the random problem (1.1) has a random solution. □

Now, we are concerned with the generalized Ulam-Hyers-Rassias stability of our problem (1.1).

Theorem 3.3. *Assume that the hypotheses* (H_1), (H_2) *and the following hypotheses hold.*

(H_3) *There exists* $\lambda_\Phi > 0$ *such that for each* $t \in I$, *and* $w \in \Omega$, *we have*

$$(I_0^\alpha \Phi)(t, w) \leq \lambda_\Phi \Phi(t, w),$$

(H_4) *There exists* $q \in C(I, [0, \infty))$ *such that for each* $t \in I$, *and* $w \in \Omega$, *we have*

$$p(t, w) \leq q(t)\Phi(t, w).$$

Then the problem (1.1) is generalized Ulam-Hyers-Rassias stable.

Proof. Consider the operator N defined in (3.1). Let u be a random solution of the inequality (2.3), and let us assume that v is a random solution of problem (1.1). Thus, we have

$$v(t, w) = \frac{\phi(w)}{\Gamma(\gamma)} t^{\gamma-1} + \int_0^t (t - s)^{\alpha-1} \frac{f(s, v(s, w), w)}{\Gamma(\alpha)} ds.$$

From the inequality (2.3) for each $t \in I$, and $w \in \Omega$, we have

$$\left| u(t, w) - \frac{\phi(w)}{\Gamma(\gamma)} t^{\gamma-1} - \int_0^t (t - s)^{\alpha-1} \frac{f(s, u(s, w), w)}{\Gamma(\alpha)} ds \right| \leq (I_0^\alpha \Phi)(t, w).$$

Set

$$q^* = \sup_{t\in I} q(t).$$

From hypotheses (H_3) and (H_4), for each $t \in I$, and $w \in \Omega$, we get

$$\begin{aligned}
|u(t,w)-v(t,w)| &\leq \left| u(t,w) - \frac{\phi(w)}{\Gamma(\gamma)}t^{\gamma-1} - \int_0^t (t-s)^{\alpha-1}\frac{f(s,u(s,w),w)}{\Gamma(\alpha)}ds \right| \\
&+ \int_0^t (t-s)^{\alpha-1}\frac{|f(s,u(s,w),w)-f(s,v(s,w),w)|}{\Gamma(\alpha)}ds \\
&\leq (I_0^\alpha \Phi)(t,w) + \int_0^t (t-s)^{\alpha-1}\frac{2q^*\Phi(s,w)}{\Gamma(\alpha)}ds \\
&\leq (I_0^\alpha \Phi)(t) + 2q^*(I_0^\alpha \Phi)(t,w) \\
&\leq [1+2q^*]\lambda_\phi \Phi(t,w) \\
&:= c_{f,\Phi}\Phi(t,w).
\end{aligned}$$

Hence, the problem (1.1) is generalized Ulam-Hyers-Rassias stable. □

In the sequel, we will use of the following Theorem.

Theorem 3.4. *[27] Let (Ω, d) be a generalized complete metric space and $\Theta : \Omega \to \Omega$ a strictly contractive operator with a Lipschitz constant $L < 1$. If there exists a nonnegative integer k such that $d(\Theta^{k+1}x, \Theta^k x) < \infty$ for any $x \in \Omega$, then the following propositions hold true:*

- (A) *The sequence $(\Theta^k x)_{n\in N}$ converges to a fixed point x^* of Θ;*
- (B) *x^* is the unique fixed point of Θ in $\Omega^* = \{y \in \Omega \mid d(\Theta^k x, y) < \infty\}$;*
- (C) *If $y \in \Omega^*$, then $d(y, x^*) \leq \frac{1}{1-L}d(y, \Theta x)$.*

Let $X = X(I, \mathbb{R})$ be the metric space, with the metric

$$d(u,v) = \sup_{t\in I} \frac{t^{1-\gamma}\|u(t,w)-v(t,w)\|}{\Phi(t,w)}.$$

Theorem 3.5. *Assume that (H_3) and the following hypothesis hold.*

(H_5) *There exists $\varphi \in C(I, [0,\infty))$ such that for each $t \in I$, $w \in \Omega$, and all $u, v \in \mathbb{R}$, we have*

$$|f(t,u,w) - f(t,v,w)| \leq t^{1-\gamma}\varphi(t)\Phi(t,w)|u-v|.$$

If

$$L := T^{1-\gamma}\varphi^*\lambda_\phi < 1, \tag{3.4}$$

where $\varphi^* = \sup_{t\in I}\varphi(t)$, *then there exists a unique random solution* u_0 *of problem (1.1), and the problem (1.1) is generalized Ulam-Hyers-Rassias stable. Furthermore, we have*

$$|u(t,w) - u_0(t,w)| \le \frac{\Phi(t,w)}{1-L},$$

for any solution u *of (2.3).*

Proof. Let N be the operator defined in (3.1). Apply Theorem 3.4, we have

$$\begin{aligned}
|(N(w)u)(t) - (N(w)v)(t)| &\le \int_0^t (t-s)^{\alpha-1}\frac{|f(s,u(s,w),w) - f(s,v(s,w),w)|}{\Gamma(\alpha)}ds \\
&\le \int_0^t (t-s)^{\alpha-1}\frac{\varphi(s)\Phi(s,w)|s^{1-\gamma}u(s,w) - s^{1-\gamma}v(s,w)|}{\Gamma(\alpha)}ds \\
&\le \int_0^t (t-s)^{\alpha-1}\frac{\varphi^*\Phi(s,w)\|u(w) - v(w)\|_C}{\Gamma(\alpha)}ds \\
&\le \varphi^*(I_0^\alpha\Phi)(t,w)\|u(w) - v(w)\|_C \\
&\le \varphi^*\lambda_\phi\Phi(t)\|u(w) - v(w)\|_C.
\end{aligned}$$

Thus

$$|t^{1-\gamma}(N(w)u)(t) - t^{1-\gamma}(N(w)v)(t)| \le T^{1-\gamma}\varphi^*\lambda_\phi\Phi(t,w)\|u(w) - v(w)\|_C.$$

Hence, we get

$$d(N(u),N(v)) \le L\|u(w) - v(w)\|_C,$$

from which we conclude the theorem. □

4 Hilfer-Hadamard fractional random differential equations

Now, we are concerned with the existence and the Ulam-Hyers-Rassias stability for problem (1.2).

Set $C := C([1,T])$. Denote the weighted space of continuous functions defined by

$$C_{\gamma,\ln}([1,T]) = \{w(t) : (\ln t)^{1-\gamma}w(t) \in C\},$$

with the norm

$$\|w\|_{C_{\gamma,\ln}} := \sup_{t\in[1,T]} |(\ln t)^{1-r} w(t)|.$$

Let us recall some definitions and properties of Hadamard fractional integration and differentiation. We refer to [22] for a more detailed analysis.

Definition 4.1. [22] (Hadamard fractional integral). The Hadamard fractional integral of order $q > 0$ for a function $g \in L^1([1,T])$, is defined as

$$({}^H I_1^q g)(x) = \frac{1}{\Gamma(q)} \int_1^x \left(\ln \frac{x}{s}\right)^{q-1} \frac{g(s)}{s} ds,$$

provided the integral exists.

Example 4.2. Let $0 < q < 1$. Then

$${}^H I_1^q \ln t = \frac{1}{\Gamma(2+q)} (\ln t)^{1+q}, \ \textit{for a.e. } t \in [0,e].$$

Set

$$\delta = x\frac{d}{dx}, \ q > 0, \ n = [q] + 1,$$

and

$$AC_\delta^n := \{u : [1,T] \to E : \delta^{n-1}[u(x)] \in AC(I)\}.$$

Analogous to the Riemann-Liouville fractional calculus, the Hadamard fractional derivative is defined in terms of the Hadamard fractional integral in the following way:

Definition 4.3. [22] (Hadamard fractional derivative). The Hadamard fractional derivative of order $q > 0$ applied to the function $w \in AC_\delta^n$ is defined as

$$({}^H D_1^q w)(x) = \delta^n ({}^H I_1^{n-q} w)(x).$$

In particular, if $q \in (0,1]$, then

$$({}^H D_1^q w)(x) = \delta ({}^H I_1^{1-q} w)(x).$$

Example 4.4. Let $0 < q < 1$. Then

$${}^H D_1^q \ln t = \frac{1}{\Gamma(2-q)} (\ln t)^{1-q}, \ \textit{for a.e. } t \in [0,e].$$

It has been proved (see e.g. Kilbas [[21], Theorem 4.8]) that in the space $L^1(I,E)$, the Hadamard fractional derivative is the left-inverse operator to the Hadamard fractional integral, i.e.

$$({}^H D_1^q)({}^H I_1^q w)(x) = w(x).$$

From Theorem 2.3 of [22], we have

$$({}^H I_1^q)({}^H D_1^q w)(x) = w(x) - \frac{({}^H I_1^{1-q} w)(1)}{\Gamma(q)}(\ln x)^{q-1}.$$

Analogous to the Hadamard fractional calculus, the Caputo-Hadamard fractional derivative is defined in the following way:

Definition 4.5. (Caputo-Hadamard fractional derivative). The Caputo-Hadamard fractional derivative of order $q>0$ applied to the function $w \in AC_\delta^n$ is defined as

$$({}^{Hc} D_1^q w)(x) = ({}^H I_1^{n-q} \delta^n w)(x).$$

In particular, if $q \in (0,1]$, then

$$({}^{Hc} D_1^q w)(x) = ({}^H I_1^{1-q} \delta w)(x).$$

From the Hadamard fractional integral, the Hilfer-Hadamard fractional derivative (introduced for the first time in [24]) is defined in the following way:

Definition 4.6. (Hilfer-Hadamard fractional derivative). Let $\alpha \in (0,1)$, $\beta \in [0,1]$, $\gamma = \alpha + \beta - \alpha\beta$, $w \in L^1(I)$, and ${}^H I_1^{(1-\alpha)(1-\beta)} w \in AC^1(I)$. The Hilfer-Hadamard fractional derivative of order α and type β applied to the function w is defined as

$$\begin{aligned} ({}^H D_1^{\alpha,\beta} w)(t) &= \left({}^H I_1^{\beta(1-\alpha)}({}^H D_1^\gamma w)\right)(t) \\ &= \left({}^H I_1^{\beta(1-\alpha)} \delta({}^H I_1^{1-\gamma} w)\right)(t); \;\; for\ a.e.\ t \in [1,T]. \end{aligned} \tag{4.1}$$

This new fractional derivative (4.1) may be viewed as interpolating the Hadamard fractional derivative and the Caputo-Hadamard fractional derivative. Indeed for $\beta = 0$ this derivative reduces to the Hadamard fractional derivative and when $\beta = 1$, we recover the Caputo-Hadamard fractional derivative.

$${}^H D_1^{\alpha,0} = {}^H D_1^\alpha, \; and \; {}^H D_1^{\alpha,1} = {}^{Hc} D_1^\alpha.$$

From Theorem 21 in [20], we concluded the following lemma

Lemma 4.7. *Let $g : I \times \mathbb{R} \times \Omega \to \mathbb{R}$ be such that $g(\cdot, u(\cdot, w), w) \in C_{\gamma,\ln}([1, T])$ for any $u(., w) \in C_{\gamma,\ln}([1, T])$. Then Then problem (1.2) is equivalent to the following volterra integral equation*

$$u(t, w) = \frac{\phi_0(w)}{\Gamma(\gamma)}(\ln t)^{\gamma-1} + ({}^H I_1^\alpha g(\cdot, u(\cdot, w), w))(t);\ w \in \Omega.$$

Definition 4.8. By a random solution of the problem (1.2) we mean a measurable function $u \in C_{\gamma,\ln}$ that satisfies the condition $({}^H I_1^{1-\gamma} u)(1^+, w) = \phi_0(w)$, and the equation $({}^H D_1^{\alpha,\beta} u)(t, w) = g(t, u(t, w), w)$ on $[1, T] \times \Omega$.

Now we give (without proof) existence and Ulam slability results for problem (1.2). The following hypotheses will be used in the sequel.

(H_1') The function g is random Carathéodory on $[1, T] \times \mathbb{R} \times \Omega$,

(H_2') There exists a measurable and bounded function $p_1 : \Omega \to L^\infty([1, T], [0, \infty))$, such that

$$|g(t, u, w)| \le \frac{p_1(t, w)|u|}{1 + |u|};\ \textit{for a.e. } t \in [1, T],\ \textit{and each } u \in \mathbb{R},\ w \in \Omega,$$

(H_3') There exists $\lambda_\Phi > 0$ such that for each $t \in [1, T]$, and $w \in \Omega$, we have

$$({}^H I_1^\alpha \Phi)(t, w) \le \lambda_\Phi \Phi(t, w),$$

(H_4') There exists $q_1 \in C(I, [0, \infty))$ such that for each $t \in I$, and $w \in \Omega$, we have

$$p_1(t, w) \le q_1(t)\Phi(t, w),$$

(H_5') There exists $\varphi_1 \in C([1, T], [0, \infty))$ such that for each $t \in [1, T],\ w \in \Omega$, and all $u, v \in \mathbb{R}$, we have

$$|g(t, u, w) - g(t, v, w)| \le (\ln t)^{1-\gamma} \varphi_1(t)\Phi(t, w)|u - v|.$$

Theorem 4.9. *Assume that the hypotheses (H_1') and (H_2') hold. Then the problem (1.2) has at least one random solution defined on $[1, T] \times \Omega$.*

Theorem 4.10. *Assume that the hypotheses $(H_1')(H_4')$ hold. Then the problem (1.2) is generalized Ulam-Hyers-Rassias stable.*

Theorem 4.11. *Assume that the hypotheses* (H_3') *and* (H_5') *hold. If*

$$L_1 := (\ln T)^{1-\gamma}\varphi_1^*\lambda_\phi < 1, \tag{4.2}$$

where $\varphi_1^* = \sup_{t\in[1,T]} \varphi_(t)$*, then there exists a unique random solution* u_1 *of problem (1.2), and the problem (1.2) is generalized Ulam-Hyers-Rassias stable. Furthermore, we have*

$$|u(t,w) - u_1(t,w)| \leq \frac{\Phi(t,w)}{1-L_1}.$$

5 An Example

As an application of our results we consider the following problem of Hilfer fractional differential equation

$$\begin{cases} (D_0^{\frac{1}{2},\frac{1}{2}}u)(t) = f(t,u(t));\; t\in[0,1], \\ (I_0^{\frac{1}{4}}u)(t)|_{t=0} = 1, \end{cases} \tag{5.1}$$

where

$$\begin{cases} f(t,u) = \dfrac{ct^{\frac{-1}{4}}\sin t}{64(1+\sqrt{t})(1+|u|)};\; t\in(0,1]\ u\in\mathbb{R}, \\ f(0,u) = 0; \qquad\qquad u\in\mathbb{R}, \end{cases}$$

and $c = \frac{9\sqrt{\pi}}{16}$. Clearly, the function f is Carathéodory.
The hypothesis (H_2) is satisfied with

$$\begin{cases} p(t) = \dfrac{ct^{\frac{-1}{4}}|\sin t|}{64(1+\sqrt{t})};\; t\in(0,1], \\ p(0) = 0. \end{cases}$$

Hence, Theorem 3.2 implies that the problem (5.1) has at least one solution defined on $[0,1]$. Also, the hypothesis (H_3) is satisfied with

$$\Phi(t) = e^3,\ and\ \lambda_\Phi = \frac{1}{\Gamma(1+\alpha)}.$$

Indeed, for each $t\in[0,1]$ we get

$$\begin{aligned} (I_0^\alpha\Phi)(t) &\leq \frac{e^3}{\Gamma(1+\alpha)} \\ &= \lambda_\Phi\Phi(t). \end{aligned}$$

Consequently, Theorem 3.3 implies that the problem (5.1) is generalized Ulam-Hyers-Rassias stable.

Acknowledgement. The authors are grateful to the referee for the careful reading of the paper.

References

[1] S. Abbas, W. A. Albarakati, M. Benchohra and J. Henderson, Existence and Ulam stabilities for Hadamard fractional integral equations with random effects, *Electron. J. Differential Equations* **2016** (2016), No. 25, pp 1-12.

[2] S. Abbas, W. Albarakati, M. Benchohra and G. M. N'Guérékata, Existence and Ulam stabilities for Hadamard fractional integral equations in Fréchet spaces, *J. Frac. Calc. Appl.* **7** (2) (2016), 1-12.

[3] S. Abbas, W.A. Albarakati, M. Benchohra and S. Sivasundaram, Dynamics and stability of Fredholm type fractional order Hadamard integral equations, *Nonlinear Stud.* **22** (4) (2015), 673-686.

[4] S. Abbas and M. Benchohra, Uniqueness and Ulam stabilities results for partial fractional differential equations with not instantaneous impulses, *Appl. Math. Comput.* **257** (2015), 190-198.

[5] S. Abbas and M. Benchohra, Existence and Ulam stability for impulsive discontinuous fractional differential inclusions in Banach Algebras, *Mediter. J. Math.* **12** (4) (2015), 1245-1264.

[6] S. Abbas and M. Benchohra, Existence and Ulam stability results for quadratic integral equations, *Libertas Math.* **35** (2)(2015), 83-93.

[7] S. Abbas, M. Benchohra, J.R. Graef and J. Henderson, *Implicit Fractional Differential and Integral Equations: Existence and Stability*, De Gruyter, Berlin, 2018.

[8] S. Abbas, M. Benchohra and G. M. N'Guérékata, *Topics in Fractional Differential Equations*, Springer, New York, 2012.

[9] S. Abbas, M. Benchohra and G. M. N'Guérékata, *Advanced Fractional Differential and Integral Equations*, Nova Science Publishers, New York, 2015.

[10] S. Abbas, M. Benchohra and A. Petrusel, Ulam stabilities for the Darboux problem for partial fractional differential inclusions via Picard Operators, *Electron. J. Qual. Theory Differ. Equ.*, **1** (2014), 1-13.

[11] S. Abbas, M. Benchohra and S. Sivasundaram, Ulam stability for partial fractional differential inclusions with multiple delay and impulses via Picard operators, *J. Nonlinear Stud.* **20** (4) (2013), 623-641.

[12] K. M. Furati and M. D. Kassim. Non-existence of global solutions for a differential equation involving Hilfer fractional derivative, *Electron. J. Differential Equations* 2013, No. 235, 10 pp.

[13] K. M. Furati, M. D. Kassim, and N. -E. Tatar, Existence and uniqueness for a problem involving Hilfer fractional derivative. *Comput. Math. Appl.* **64** (2012), 1616-1626.

[14] R. Hilfer, *Applications of Fractional Calculus in Physics*, World Scientific, Singapore, 2000.

[15] D.H. Hyers, On the stability of the linear functional equation, *Proc. Nat. Acad. Sci.* **27** (1941), 222-224.

[16] S. Itoh, Random fixed point theorems with applications to random differential equations in Banach spaces, *J. Math. Anal. Appl.* **67** (1979), 261-273.

[17] S.-M. Jung, *Hyers-Ulam-Rassias Stability of Functional Equations in Mathematical Analysis*, Hadronic Press, Palm Harbor, 2001.

[18] S.-M. Jung, *Hyers-Ulam-Rassias Stability of Functional Equations in Nonlinear Analysis*, Springer, New York, 2011.

[19] R. Kamocki and C. Obczńnski, On fractional Cauchy-type problems containing Hilfer's derivative, *Electron. J. Qual. Theory Differ. Equ.*, 2016, No. 50, 1-12.

[20] M. D. Kassim and N.-e. Tatar, Well-posedness and stability for a differential problem with Hilfer-Hadamard fractional derivative, *Abstr. Appl. Anal.* Volume 2013, Article ID 605029, 12 pages, 2013.

[21] A. A. Kilbas, Hadamard-type fractional calculus, *J. Korean Math. Soc.* **38** (6) (2001), 1191-1204.

[22] A. A. Kilbas, H. M. Srivastava and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, Elsevier Science B.V., Amsterdam, 2006.

[23] T.P. Petru, A. Petrusel. J.-C. Yao, Ulam-Hyers stability for operatorial equations and inclusions via nonself operators, *Taiwanese J. Math.* **15** (2011), 2169-2193.

[24] M. D. Qassim, K. M. Furati, and N.-E. Tatar, On a differential equation involving Hilfer-Hadamard fractional derivative, *Abstr. Appl. Anal.* vol. 2012, Article ID 391062, 17 pages, 2012.

[25] Th.M. Rassias, On the stability of linear mappings in Banach spaces, *Proc. Amer. Math. Soc.* **72** (1978), 297-300.

[26] I. A. Rus, Ulam stability of ordinary differential equations, *Studia Univ. Babes-Bolyai, Math.* **LIV** (4)(2009), 125-133.

[27] I. A. Rus, Remarks on Ulam stability of the operatorial equations, *Fixed Point Theory* **10** (2009), 305-320.

[28] S. G. Samko, A. A. Kilbas and O. I. Marichev, *Fractional Integrals and Derivatives. Theory and Applications*, Gordon and Breach, Amsterdam, 1987, Engl. Trans. from the Russian.

[29] V. E. Tarasov, *Fractional Dynamics: Application of Fractional Calculus to Dynamics of Particles, Fields and Media*, Springer, Heidelberg; Higher Education Press, Beijing, 2010.

[30] Ž. Tomovski, R. Hilfer and H.M. Srivastava, Fractional and operational calculus with generalized fractional derivative operators and Mittag-Leffler type functions, *Integral Transforms Spec. Funct.* **21** (11) (2010), 797-814.

[31] S.M. Ulam, *A Collection of Mathematical Problems*, Interscience Publishers, New York, 1968.

[32] J.-R. Wang, and Y. Zhang, Nonlocal initial value problems for differential equations with Hilfer fractional derivative. *Appl. Math. Comput.* **266** (2015), 850-859.

[33] Y. Zhou, *Basic Theory of Fractional Differential Equations*, World Scientific, Singapore, 2014.

[34] Y. Zhou, *Fractional Evolution Equations and Inclusions : Analysis and Control*, Elsevier Science, 2016.

Saïd Abbas
Laboratory of Mathematics, Geometry, Analysis, Control and Applications
Tahar Moulay University of Saïda, P.O. Box 138, EN-Nasr, 20000 Saïda, Algeria

E-mail: said.abbas@univ-saida.dz, abbasmsaid@yahoo.fr

Mouffak Benchohra
Laboratory of Mathematics, Djillali Liabes University of Sidi Bel-Abbes, P.O. Box 89, Sidi Bel-Abbès 22000, Algeria
E-mail: benchohra@univ-sba.dz

Naima Hamidi
Laboratory of Mathematics, Djillali Liabes University of Sidi Bel-Abbes, P.O. Box 89, Sidi Bel-Abbès 22000, Algeria
E-mail: hamidi.naima@yahoo.fr